Comptes Rendus (Doklady) de l'Académie des Sciences de l'URSS
1937. Volume XVII, № 9

MATHÉMATIQUES

SUR UNE GÉNÉRALISATION DE L'INÉGALITÉ DE MINKOVSKI

Par A. GELFOND

(Présenté par I. M. Vinogradow, de l'Académie, le 1 XI 1937)

Je me propose de donner dans cette note une généralisation d'un théorème bien connu de Minkovski: on peut toujours résoudre en nombres entiers $x_1, x_2, \ldots x_n$ un système d'inégalités:

$$|\xi_\nu| \leqslant t_\nu, \quad \nu = 1, 2, \ldots n, \quad t_1 t_2 \ldots t_n = \Delta, \tag{1}$$

où $\xi_\nu = \sum_{\mu=1}^{n} \alpha_{\nu,\mu} x_\mu$ et $\Delta = |\alpha_{\nu,\mu}|_1^n > 0$ est le déterminant du système de formes linéaires.

Pour généraliser ce théorème de Minkovski je me sers d'une formule très intéressante obtenue par Siegel:

$$\Delta \sum{}' \prod_{\nu=1}^{n} (t_\nu - |\xi_\nu|) =$$

$$= (t_1 \ldots t_n)^2 \sum_{y_1 y_2 \ldots y_n = -\infty}^{\infty} \prod_{\nu=1}^{n} \left[\frac{\sin \pi (A_{\nu,1} y_1 + \ldots + A_{\nu,n} y_n) t_\nu}{\pi (A_{\nu,1} y_1 + \ldots + A_{\nu,n} y_n) t_\nu} \right]^2,$$

$$\xi_\nu = \sum_{\mu=1}^{n} \alpha_{\nu,\mu} x_\mu, \ \Delta = |\alpha_{\nu,\mu}|_1^n > 0, \ x_\mu = \sum_{\nu=1}^{n} A_{\nu,\mu} \xi_\nu, \tag{2}$$

où $A_{\nu,\mu}$ sont les coefficients de la transformation inverse, tandis que la somme dans le premier membre est étendue à toutes les solutions en nombres entiers $x_1, x_2, \ldots x_n$ du système des inégalités $|\xi_\nu| < t_\nu$, $\nu = 1, 2, \ldots n$.

C. Siegel utilise cette formule pour démontrer le théorème en question de Minkovski. En rendant plus exacte la démonstration de Siegel on peut obtenir un théorème un peu plus général que le théorème de Minkovski.

Désignons par $\omega(t_1, t_2, \ldots t_n)$ le nombre de solutions (sans compter la solution $0, 0, \ldots 0$) en nombres entiers $x_1, x_2, \ldots x_n$ du système des inégalités:

$$|\xi_\nu| \leqslant t_\nu, \ \xi_\nu = \sum_{\mu=1}^{n} \alpha_{\nu,\mu} x_\mu, \ \nu = 1, 2, \ldots n. \tag{3}$$

et par $\Omega(z_1, z_2, \ldots z_n)$ le nombre de solutions (sans compter $0, 0, \ldots 0$) en nombres entiers $y_1, y_2, \ldots y_n$ du système des inégalités:

$$|\eta_\nu| < z_\nu, \ \eta_\nu = \sum_{\mu=1}^{n} A_{\mu,\nu} y_\mu, \ \nu = 1, 2, \ldots n, \tag{4}$$

où les nombres $A_{\mu,\nu}$ sont les coefficients de la transformation $x_\mu = \sum_{\nu=1}^{n} A_{\mu,\nu} \xi_\nu$. Avec cette notation on a l'inégalité

$$\Delta[1 + \omega(t_1, \ldots t_n)] > t_1 t_2 \ldots t_n \left[1 + \frac{\sin^2 \lambda\pi}{\lambda^2 \pi^2} \Omega\left(\frac{\lambda}{t_1}, \frac{\lambda}{t_2}, \ldots \frac{\lambda}{t_n}\right)\right], \tag{5}$$

où λ est un nombre arbitraire $0 < \lambda < 1$, et $\Delta = |\alpha_{\nu,\mu}|_1^n > 0$. On obtient cette inégalité immédiatement de la formule (2), si l'on rejette dans tous les produits du premier membre $|\xi_\nu|$, $\nu = 1, 2, \ldots n$ et dans ceux du second membre tous les termes excepté ceux, pour lesquels toutes les inégalités (4) ont lieu pour $z_1 = \frac{\lambda}{t_1}$, $z_2 = \frac{\lambda}{t_2}$, $\ldots z_n = \frac{\lambda}{t_n}$. L'inégalité (5) donne comme conséquence le théorème généralisé de Minkovski.

Théorème. *Il existe un système de nombres entiers $x_1, x_2, \ldots x_n$ qui ne sont pas nuls simultanément, et qui satisfont au système des inégalités:*

$$\left|\sum_{\mu=1}^{n} \alpha_{\nu,\mu} x_\mu\right| \leqslant t_\nu, \ \nu = 1, 2, \ldots n, \ t_1 t_2 \ldots t_n \geqslant \frac{\Delta}{1 + \frac{\sin^2 \pi\lambda}{(\pi\lambda)^2} \Omega}, \tag{6}$$

où les valeurs Δ, λ et $\Omega = \Omega\left(\frac{\lambda}{t_1}, \frac{\lambda}{t_2}, \ldots \frac{\lambda}{t_n}\right)$ ont le sens précédemment indiqué.

L'inégalité (5), donnant une relation entre le nombre de solutions des inégalités (3) et (4) présente un certain intérêt parce qu'on en déduit immédiatement plusieurs théorèmes de la théorie des approximations diophantiques, par exemple le théorème démontré par A. Khintschine: s'il existe une infinité de systèmes de nombres entiers $x_1, x_2, \ldots x_n$ vérifiant, pour $c_1 > 0$ et $\varepsilon > 0$ constants, l'inégalité

$$\left|\sum_{k=1}^{n} x_k \alpha_k + x_{n+1}\right| < c_1 H^{-(n+\varepsilon)}, \ H = \max_{1 \leqslant k \leqslant n+1} |x_k|, \tag{7}$$

alors il existe aussi une infinité de systèmes de nombres entiers q, p_1, $p_2 \ldots p_n$ qui satisfont, pour une certaine constante $c_2 > 0$ et pour le même ε à l'ensemble des inégalités:

$$\left|\alpha_k - \frac{p_k}{q}\right| < c_2 q^{-1 - \frac{n+\varepsilon}{n^2 + \varepsilon(n-1)}}, \ k = 1, 2, \ldots n. \tag{8}$$

Pour démontrer ce théorème posons

$$\xi_\nu = x_\nu + \alpha_\nu x_{n+1}, \ \nu = 1, \ldots n, \ \xi_{n+1} = x_{n+1};$$

$$t_1 = t_2 = \ldots = t_n = t_0, \ t_{n+1} = T, \ t_0 = c_2 T^{-\frac{n+\varepsilon}{n^2 + \varepsilon(n-1)}}; \tag{9}$$

$$\lambda = \frac{1}{2}.$$

Nous verrons alors que $\Delta = 1$ et que

$$x_\mu = \xi_\mu - \alpha_\mu \xi_{n+1}, \ x_{n+1} = \xi_{n+1} \ (\mu = 1, \ldots n). \tag{10}$$

Supposons que pour un H donné l'inégalité (7) est vérifiée. En posant alors

$$T = (2C_2)^{\frac{n^2+\varepsilon(n-1)}{n+\varepsilon}} H^{\frac{n^2+\varepsilon(n-1)}{n}}, \tag{11}$$

on voit facilement que, pour un C_2 assez grand, et indépendant de H, toutes les conditions de notre théorème précédent sont remplies. En effet, dans ce cas le nombre $\Omega\left(\frac{\lambda}{t_1}, \frac{\lambda}{t_2}, \ldots \frac{\lambda}{t_{n+1}}\right)$ présente le nombre de solutions en nombres entiers des inégalités:

$$|y_i| < \frac{1}{2t_0}, \; i = 1, 2, \ldots n, \; \left| \sum_{k=1}^{n} y_k \alpha_k + y_{n+1} \right| < \frac{1}{2T}. \tag{12}$$

Comme chaque solution de l'inégalité (7) est, en même temps, une solution des inégalités (12) et, d'ailleurs, les nombres multiples des nombres de cette solution satisfont à l'inégalité (12), on voit, en faisant un calcul simple et en choisissant t_0 et T comme il a été indiqué, que Ω surpasse, supposant C assez grand,

$$\frac{1}{4} c_1^{-1} (2c_2)^{-n} T^{\frac{\varepsilon}{n^2+\varepsilon(n-1)}}, \; c_2 = c_2(n, c_1, \varepsilon). \tag{13}$$

Le théorème que nous démontrons en résulte immédiatement.

Par la même méthode on peut démontrer le théorème de A. Khintschine dans la direction inverse. D'une façon générale, notre généralisation du théorème de Minkovski présente un intérêt dans le cas où l'on réussit de calculer ou bien de trouver une borne inférieure pour le nombre $\Omega\left(\frac{\lambda}{t_1}, \ldots \frac{\lambda}{t_n}\right)$.

Manuscrit reçu
le 1 XI 1937.

Comptes Rendus (Doklady) de l'Académie des Sciences de l'URSS
1937. Volume XVII, № 9

MATHÉMATIQUES

SUR UNE CLASSE D'ÉQUATIONS INTÉGRODIFFÉRENTIELLES À PLUSIEURS VARIABLES INDÉPENDANTES

Par S. Sobolev (Soboleff), membre correspondant de l'Académie

I. L'objet de cet article est l'étude de l'équation intégrodifférentielle:

$$u(x_1^{(0)}, \dots x_n^{(0)}, t^{(0)}) = f(x_1^{(0)}, \dots x_n^{(0)}, t^{(0)}) +$$

$$+ \overbrace{\int \dots \int}^{n+1} \sum_{p_0, \dots p_n} K_{p_0, \dots p_n}(x_1^{(0)}, \dots x_n^{(0)}, t^{(0)}, x_1^{(1)}, \dots x_n^{(1)}, t^{(1)})$$

$$\frac{\partial^p u(x_1^{(1)}, \dots x_n^{(1)}, t^{(1)})}{\partial t^{(1)p_0} \partial x_1^{(1)p_1} \dots \partial x_n^{(1)p_n}} dx_1^{(1)} \dots dx_n^{(1)}, \qquad (1)$$

$$0 \leqslant t^{(1)} \leqslant t^{(0)} - r^{(01)}$$

$$r^{(01)} = \sqrt{\sum_{i=1}^{n} (x_i^{(0)} - x_i^{(1)})^2}.$$

Le domaine de l'intégration est un cône dans l'espace à $n+1$ dimensions ayant pour base le plan $t^{(1)} = 0$. L'équation de ce cône est:

$$t^{(1)} = t^{(0)} - r^{(01)}. \qquad (2)$$

La somme dans le second membre de (1) ne contient qu'un nombre fini de termes. Nous écrirons l'équation (1) sous la forme

$$u = f + Lu. \qquad (3)$$

Supposons que le noyau $K_{p_0, \dots p_n}(x_1^{(0)}, \dots x_n^{(0)}, t^{(0)}, x_1^{(1)}, \dots x_n^{(1)}, t^{(1)})$ et la fonction f satisfassent quelques conditions restrictives.

Considérons chaque noyau $K_{p_0, \dots p_n}$ comme fonction du point variable $x_1^{(0)}, x_2^{(0)}, \dots x_n^{(0)}, t^{(0)}$ (le sommet du cône) et des differences:

$$\xi_0^{(01)} = t^{(0)} - t^{(1)}, \quad \xi_i^{(01)} = x_i^{(0)} - x_i^{(1)}.$$

Soit

$$K_{p_0, \dots p_n}(x_1^{(0)}, \dots x_n^{(0)}, t^{(0)}, x_1^{(1)}, x_2^{(1)}, \dots x_n^{(1)}, t^{(1)}) =$$
$$= \overline{K}_{p_0, \dots p_n}(x_1^{(0)}, \dots x_n^{(0)}, t^{(0)}, \xi_1^{(01)}, \xi_2^{(01)}, \dots \xi_n^{(01)}, \xi_0^{(01)}). \qquad (4)$$

Soit encore

$$\frac{\xi_i^{(01)}}{\xi_0^{(01)}} = \eta^{(01)}; \quad \sqrt{\sum_{i=1}^{n} \eta_i^{(01)2}} = \eta^{(01)},$$

$$\eta_i^{(01)} = \eta^{(01)} \sin\vartheta_1^{(01)} \sin\vartheta_2^{(01)} \ldots \sin\vartheta_{i-1}^{(01)} \cos\vartheta_i^{(01)} \quad (i < n-1),$$
$$\eta_{n-1}^{(01)} = \eta^{(01)} \sin\vartheta_1^{(01)} \ldots \sin\vartheta_{n-2}^{(01)} \cos\varphi^{(01)},$$
$$\eta_n^{(01)} = \eta^{(01)} \sin\vartheta_1^{(01)} \ldots \sin\vartheta_{n-2}^{(01)} \sin\varphi^{(01)},$$

et soit:

$$K_{p_0,\ldots p_n}(x_1^{(0)},\ldots x_n^{(0)}, t^{(0)}; x_1^{(1)},\ldots x_n^{(1)}, t^{(1)}) =$$
$$= \overline{\overline{K}}_{p_0,\ldots p_n}(x_1^{(0)},\ldots x_n^{(0)}, t^{(0)}; \eta^{(01)}, \vartheta_1^{(01)},\ldots \vartheta_{n-2}^{(01)}, \varphi^{(01)}, \xi_0^{(01)}). \qquad (5)$$

Nous dirons que le noyau $\overline{\overline{K}}_{p_0,\ldots p_n}$ est un noyau régulier de degrés $\lambda_{p_0,\ldots p_n}$ et $\mu_{p_0,\ldots p_n}$ si les deux inégalités suivantes sont remplies:

pour $\eta^{(01)} \leqslant \frac{2}{3}$:

$$\left| \frac{\partial^l \overline{\overline{K}}_{p_0,\ldots p_n}}{\partial t^{(0)\gamma_0} \partial x_1^{(0)\gamma_1} \ldots \partial x_n^{(0)\gamma_n} \partial \xi_0^{(01)\varepsilon_0} \partial \xi_1^{(01)\varepsilon_1} \ldots \partial \xi_n^{(01)\varepsilon_n}} \right| \leqslant M \xi_0^{(01)\lambda_{p_0,\ldots p_n} - \sum_{j=0}^{n} \varepsilon_j};$$

et pour $\eta^{(01)} \geqslant \frac{1}{3}$: (6)

$$\left| \frac{\partial^l \overline{\overline{K}}_{p_0,\ldots p_n}}{\partial t^{(0)\gamma_0} \partial x_1^{(0)\gamma_1} \ldots \partial x_n^{(0)\gamma_n} \partial \xi_0^{(01)\varepsilon_0} \partial \eta^{(01)\varepsilon_1} \partial \vartheta_1^{(01)\varepsilon_2} \ldots \partial \varphi^{(01)\varepsilon_n}} \right| \leqslant$$
$$\leqslant M \xi_0^{(01)\lambda_{p_0,\ldots p_n} - \varepsilon_0} (1-\eta^{(01)})^{\mu_{p_0,\ldots p_n} - \varepsilon_1};$$

où

$$\lambda_{p_0,\ldots p_n} > -n-1 \text{ et } \mu_{p_0,\ldots p_n} > -1. \qquad (7)$$

On peut établir que les inégalités (6) entraînent deux autres dans lesquelles les rôles des points $x_1^{(0)},\ldots x_n^{(0)}, t^{(0)}$ et $x_1^{(1)},\ldots x_n^{(1)}, t^{(1)}$ sont changés.

En étudiant l'opération intégrodifférentielle L nous introduisons deux nombres que nous appellerons indices de régularité.

Considérons pour chaque noyau $K_{p_0,\ldots p_n}$ le nombre

$$\lambda_{p_0,\ldots p_n} + n + 1 - p. \qquad (8)$$

Nous désignerons par ρ_1 le minimum de tous ces nombres et nous l'appellerons «premier indice de régularité de l'opération L».

D'une façon analogue, le second indice de régularité ρ_2 est le plus petit parmi les nombres:

$$\mu_{p_0,\ldots p_n} + \frac{n+1}{2} - l. \qquad (9)$$

Soit s le maximum de p_0 dans l'opération L.

Convenons de dire que la fonction $f(x_1^{(0)},\ldots x_n^{(0)}, t^{(0)})$ appartient à la classe ε_1 pour l'opération L si elle admet un certain nombre de dérivées vérifiant l'inégalité:

$$\left| \frac{\partial^m f}{\partial t^{(0)\alpha_0} \partial x_1^{(0)\alpha_1} \ldots \partial x_n^{(0)\alpha_n}} \right| \leqslant A t^{(0)s - \alpha_0 + \varepsilon_1}. \qquad (10)$$

Convenons de dire que le terme $f(x_1^{(0)},\ldots x_n^{(0)}, t^{(0)})$ est régulier pour l'équation (1) ou (3) si les conditions suivantes sont remplies:

a) On peut appliquer l'opération L à la fonction f un certain nombre de fois, c'est-à-dire, on peut construire les fonctions:

$$Lf,\ L^2f,\ \ldots\ L^qf.$$

b) La fonction L^qf appartient à la classe ε_1 où $\varepsilon_1 > -1$.

Le résultat principal du présent article est le théorème suivant:

Théorème. *Si dans l'équation (3) tous les noyaux sont réguliers, les deux indices de régularité sont positifs et le terme indépendant de u est régulier; cette équation peut être résolue par la méthode des approximations successives:*

$$u = f + \sum_{j=1}^{\infty} L^j f,$$

et cette solution est unique.

La démonstration de ce théorème s'appuit sur quelques lemmes.

Lemme I. *Soit f une fonction qui appartient à la classe ε_1 pour l'opération L. Alors la fonction Lf (qui admet, peut être, moins de dérivées) appartient à la classe $\rho_1 + \varepsilon_1$, où ρ_1 est le premier indice de régularité de l'opération L.*

Voici une conséquence de ce lemme:

Lemme II. *Si le premier indice de régularité ρ_1 de l'opération L est positif, et la fonction f appartient à la classe $\rho_1 > -1$ on peut choisir un nombre entier k tel que la fonction*

$$L^k f$$

s'annule ainsi qu'un certain nombre de ses dérivées (donné d'avance) pour $t^{(0)} = 0$. (Il faut certainement supposer que f admet un nombre assez grand de dérivées.)

Ces deux lemmes permettent de transformer l'équation (3) en une autre, dans laquelle le terme indépendant de u s'annule pour $t^{(0)} = 0$, *ainsi que ses dérivées jusqu'à l'ordre donné. Dans ce but il faut poser:*

$$u_1 = u - f - \sum_{j=1}^{k-1} L^j f. \tag{13}$$

Alors l'équation (3) est remplacée par l'équation équivalente:

$$u_1 = L^k f + L u_1. \tag{14}$$

Le lemme suivant joue un rôle important.

Lemme III. *Si le noyau $K^{(02)}(x_1^{(0)}, \ldots x_n^{(0)}, t^{(0)}, x_1^{(2)}, \ldots x_n^{(2)}, t^{(2)})$ est régulier et de degrés $\lambda^{(02)}$ et $\mu^{(02)}$ et le noyau $K^{(21)}(x_1^{(2)}, \ldots x_n^{(2)}, t^{(2)}; x_1^{(1)}, \ldots x_n^{(1)}, t^{(1)})$ est régulier et de degrés $\lambda^{(21)}$ et $\mu^{(21)}$, le noyau itéré:*

$$K^{(01)}(x_1^{(0)}, \ldots x_n^{(0)}, t^{(0)}; x_1^{(1)}, \ldots x_n^{(1)}, t^{(1)} =$$

$$= \overbrace{\int \ldots \int}^{n+1} K^{(02)}(x_1^{(0)}, \ldots x_n^{(0)}, t^{(0)}; x_1^{(2)}, \ldots x_n^{(2)}, t^{(2)}) \cdot \tag{15}$$
$$t^{(1)} + r^{(21)} \leqslant t^{(2)} \leqslant t^{(0)} - r^{(02)}$$
$$K^{(21)}(x_1^{(2)}, \ldots x_n^{(2)}, t^{(2)}; x_1^{(1)}, \ldots x_n^{(1)}, t^{(1)})\, dx_1^{(2)} \ldots dx_n^{(2)}\, dt^{(2)},$$

est à son tour régulier et de degrés

$$\lambda^{(01)} = \lambda^{(02)} + \lambda^{(21)} + n + 1$$

et (16)

$$\mu^{(01)} = \min\left(\mu^{(02)} + \mu^{(21)} + \frac{n+1}{2},\ \lambda^{(02)} + \mu^{(21)} + n + 1,\ \lambda^{(21)} + \mu^{(02)} + n + 1\right).$$

De ce lemme il résulte:

Lemme IV. *Soit L_1 une opération intégrodifférentielle dont les indices de régularité sont $\rho_1^{(1)}$ et $\rho_2^{(1)}$ et L_2 une autre opération aux indices $\rho_1^{(2)}$ et $\rho_2^{(2)}$. Alors $L_3 = L_1 L_2$ sera aussi une opération du type considéré dont les indices $\rho_1^{(3)}$ et $\rho_2^{(3)}$ sont définis par les formules:*

$$\rho_1^{(3)} = \rho_1^{(1)} + \rho_1^{(2)},$$
$$\rho_2^{(3)} = \min\left(\rho_2^{(1)} + \rho_2^{(2)}, \rho_2^{(1)} + \rho_1^{(2)}, \rho_2^{(2)} + \rho_1^{(1)}\right). \tag{17}$$

Énoncons encore quelques lemmes:

Lemme V. *Pour les fonctions f de classe $\varepsilon_1 > N$, où N est un nombre assez grand, l'opération L dont les indices satisfont aux inégalités:*

$$\rho_1 > -1, \quad \rho_2 > \frac{n-1}{2}, \tag{18}$$

est équivalente à une opération intégrale. Autrement dit

$$Lf = \overbrace{\int \cdots \int}^{n+1} \mathbf{K}(x_1^{(0)}, \ldots x_n^{(0)}, t^{(0)}; x_1^{(1)}, \ldots x_n^{(1)}, t^{(1)}) \times$$
$$0 \leqslant t^{(1)} \leqslant t^{(0)} - r^{(01)}$$
$$\times f(x_1^{(1)}, \ldots x_n^{(1)}, t^{(1)})\, dx_1^{(1)} \ldots dx_n^{(1)}\, dt^{(1)}. \tag{19}$$

On démontre ce lemme en intégrant par parties:

Lemme VI. *Si les deux indices de régularité de l'opération L sont positifs il existe une puissance L^h de cette opération qui est équivalente à une opération intégrale pour les fonctions f de classe $\varepsilon_1 > N$.*

Les lemmes précédents démontrent que nous pouvons dans le cas considéré remplacer l'équation (3) par une équation intégrale du type de Volterra à l'aide d'un nombre fini d'itérations. Nous aurons:

$$u_1(x_1^{(0)}, \ldots x_n^{(0)}, t^{(0)}) = \sum_{j=k}^{h+h-1} L^j f +$$
$$+ \overbrace{\int \cdots \int}^{n+1} \mathbf{K}(x_1^{(0)}, \ldots x_n^{(0)}, t^{(0)}; x_1^{(1)}, \ldots x_n^{(1)}, t^{(1)}) \times$$
$$0 \leqslant t^{(1)} \leqslant t^{(0)} - r^{(01)}$$
$$\times u_1(x_1^{(1)}, \ldots x_n^{(1)}, t^{(1)})\, dx_1^{(1)} \ldots dx_n^{(1)}\, dt^{(1)}. \tag{20}$$

L'équation (20) peut être résolue au moyen de la méthode des approximations successives. Les lemmes énoncés permettent d'achever la démonstration du théorème fondamental.

Il est utile de remarquer, que le terme indépendant de u dans l'équation (1) sera régulier, par exemple, si p_0 est constant et si f a des dérivées bornées.

Dans un article en préparation nous allons démontrer que la condition $\rho_2 > 0$ peut être remplacée quelquefois par la condition $\rho_2 = 0$.

Nous allons démontrer aussi que les équations pour lesquelles $\rho_1 = 0$ où $\rho_2 < 0$ n'admettent pas de solutions dans le cas général.

La démonstration complète de tout ce qui est énoncé ici sera donnée dans le Bull. Acad. Sci. URSS (classe des sciences mathém.).

Institut mathématique V. A. Stekloff.
Académie des Sciences de l'URSS.

Manuscrit reçu
le 2. XI. 1937.

Comptes Rendus (Doklady) de l'Académie des Sciences de l'URSS
1937. Volume XVII, № 9

MATHEMATIK

ÜBER DIE BESTE ANNÄHERUNG EINER KLASSE STETIGER PERIODISCHER FUNKTIONEN

Von N. ACHYESER

(Vorgelegt von S. N. Bernstein, Mitglied der Akademie, am 14. XI. 1937)

Es seien gegeben die beiden Differentialoperatoren:

$$L = (D^2 + \alpha_1)(D^2 + \alpha_2) \ldots (D^2 + \alpha_r),$$
$$M = D(D^2 + \alpha_1)(D^2 + \alpha_2) \ldots (D^2 + \alpha_r),$$

wo $D = \frac{d}{dt}$ und $\alpha_1, \alpha_2, \ldots, \alpha_r$ reelle Zahlen sind, welche wir zur Vereinfachung voneinander verschieden und im Falle des Operators M auch von Null verschieden voraussetzen wollen.

Es sei ferner H_L (bzw. H_M) die Klasse aller genügend oft differentierbarer komplexwertiger Funktionen $x(t)$ mit der Periode 2π, für welche $f(t) = L[x]$ (bzw. $g(t) = M[x]$) fast überall dem absoluten Betrage nach $\leqslant 1$ bleibt.

Ist $\mathcal{E}_m[x]$ die übliche Bezeichnung der besten gleichmässigen Annäherung von $x(t)$ vermöge trigonometrischer Summen der Ordnung $\leqslant m$, so besteht der folgende

Satz. *Falls die Ungleichung*

$$n^2 > \operatorname{Max}\{\alpha_1, \alpha_2, \ldots, \alpha_r\}$$

erfüllt ist, so ist

$$\mathcal{E}_{n-1}\{H_L\} = \sup_{x \subset H_L} \mathcal{E}_{n-1}[x] = \left| \sum_{k=1}^{r} \frac{1}{\alpha_k \omega'(\alpha_k)} \frac{2\sin^2 \frac{\pi\sqrt{\alpha_k}}{4n}}{\cos \frac{\pi\sqrt{\alpha_k}}{2n}} \right| =$$

$$= \left| \frac{E_{2r}}{(2r)!}\left(\frac{\pi}{2n}\right)^{2r} - \frac{E_{2r+2}}{(2r+2)!}\left(\frac{\pi}{2n}\right)^{2r+2} \sum_{k=1}^{r} \frac{\alpha_k'}{\omega'(\alpha_k)} + \ldots \right|,$$

$$\mathcal{E}_{n-1}\{H_M\} = \sup_{y \subset H_M} \mathcal{E}_{n-1}[y] = \left| \sum_{k=0}^{r} \frac{1}{\sqrt{\alpha_k}\,\Omega'(\alpha_k)} \operatorname{tg} \frac{\pi\sqrt{\alpha_k}}{2n} \right| =$$

$$= \left| \frac{C_{2r+1}}{(2r+1)!}\left(\frac{\pi}{2n}\right)^{2r+1} - \frac{C_{2r+3}}{(2r+3)!}\left(\frac{\pi}{2n}\right)^{2r+3} \sum_{k=0}^{r} \frac{\alpha_k^{r+1}}{\Omega'(\alpha_k)} + \ldots \right|,$$

wo

$$\omega(z) = (z - \alpha_1)(z - \alpha_2) \ldots (z - \alpha_r),$$
$$\Omega(z) = (z - \alpha_0)(z - \alpha_1) \ldots (z - \alpha_r) \qquad (\alpha_0 = 0),$$

gesetzt ist, und die Zahlen E_k, C_k vermöge der Entwicklungen (1)

$$\operatorname{tg} z = \sum_{k=0}^{\infty} (-1)^{k+1} \frac{z^{2k+1}}{(2k+1)!} C_{2k+1}$$

$$\frac{1}{\cos z} = \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k}}{(2k)!} E_{2k},$$

erklärt sind (E_k sind Eulersche Zahlen).

Die extremale Gleichung $\mathcal{E}_{n-1}[x] = \mathcal{E}_{n-1}\{H_L\}$, wo $x(t) \subset H_L$ (bzw. die Gleichung $\mathcal{E}_{n-1}[y] = \mathcal{E}_{n-1}\{H_M\}$, wo $y(t) \subset H_M$), besteht dabei dann und nur dann, wenn $L[x]$ (bzw. $M[y]$) fast überall der Funktion $e^{i\alpha}$ sign $\left(\sin n\,(t-\beta)\right)$ gleich ist (α, β bedeuten reelle Konstanten).

Beweis. Es sei $x(t) \subset H_L$; dann ist

$$x(t) = s_{n-1}(t) + \frac{(-1)^r}{\pi} \int_0^{\pi} \left\{ \sum_{j=1}^{r} \frac{1}{\omega'(\alpha_j)} \sum_{k=n}^{\infty}{}' \frac{(-1)^k \cos ku}{k^2 - \alpha_j} - \sum_{k=0}^{n-1} A_k \cos ku \right\} \left\{ f(t+\pi-u) + f(t+\pi+u) \right\} du,$$

wobei $f(t) = L[x]$ ist, A_k $(k = 0, 1, \ldots, n-1)$ beliebige reelle Zahlen bezeichnen und $s_{n-1}(t)$ eine von den Zahlen A_k abhängige trigonometrische Summe der Ordnung $\leqslant n-1$ bedeutet. Alles wird jetzt zurückgeführt auf den Beweis der Tatsache, dass der Ausdruck

$$\varphi(u) = \sum_{j=1}^{r} \frac{1}{\omega'(\alpha_j)} \sum_{k=n}^{\infty}{}' \frac{(-1)^k \cos ku}{k^2 - \alpha_j} - \sum_{k=0}^{n-1} A_k \cos ku$$

für $n^2 > \operatorname{Max}\{\alpha_1, \alpha_2, \ldots, \alpha_r\}$ im Intervalle $0 < u < \pi$ immer nicht mehr als n Nullstellen hat.

Zu diesem Zwecke dienen die Entwicklungen $(-\pi < u < \pi)$:

$$\sum_{k=1}^{\infty}{}' \frac{(-1)^k \cos ku}{k^2 - \mu^2} = \frac{1}{2\mu^2} - \frac{\pi \cos \mu u}{2\mu \sin \mu\pi},$$

$$\cos \mu u = 1 - \frac{\mu^2}{2!}\left(2 \sin \frac{u}{2}\right)^2 + \frac{\mu^2(\mu^2-1)}{4!}\left(2 \sin \frac{u}{2}\right)^4 - \frac{\mu^2(\mu^2-1)(\mu^2-2^2)}{6!}\left(2 \sin \frac{u}{2}\right)^6 + \ldots$$

wo μ eine beliebige reelle oder rein imaginäre Zahl sein kann (in der ersten Reihe müssen zunächst die ganzen Werte von μ ausgeschlossen werden).

Auf Grund dieser Entwicklungen ist

$$\varphi(u) = \sum_{k=n}^{\infty} \frac{(2k-2)!!}{(2k-1)!!} \frac{1}{k} \sin^{2k} \frac{u}{2} \sum_{j=1}^{r} \frac{1}{\omega'(\alpha_j) \prod_{m=k}^{\infty} \left(1 - \frac{\alpha_j}{m^2}\right)} - \sum_{k=0}^{n-1} A_k^* \sin^{2k} \frac{u}{2},$$

und betrachtet man $\sin^2 \frac{u}{2}$ als unabhängige Variable, so genügt es zu zeigen, dass alle

$$Q_k = \sum_{j=1}^{r} \frac{1}{\omega'(\alpha_j) \prod_{m=k}^{\infty} \left(1 - \frac{\alpha_j}{m^2}\right)}$$

für $k^2 \geqslant n^2 > \operatorname{Max}\{\alpha_1, \alpha_2, \ldots, \alpha_r\}$ ein und dasselbe Vorzeichen haben. Nun ist aber Q_k gleich dem Werte der $(r-1)$-ten Ableitung von

$$\frac{1}{(r-1)!} \cdot \frac{1}{\prod_{m=k}^{\infty} \left(1 - \frac{x}{m^2}\right)}$$

in einem Punkte ξ_k, der zwischen $\alpha_1, \alpha_2 \ldots, \alpha_r$ liegt und also kleiner als n^2 ist, so dass, wie man leicht sieht, alle $Q_k > 0$ $(k \geqslant n)$, w. z. b. w.*

Ähnliche Verhältnisse hat man im Falle des Operators M.

Die extremale Funktion $\operatorname{sign}(\sin nt)$ bleibt dieselbe, auch wenn charakteristische Polynome $\omega(z)$, $\Omega(z)$ vielfache Wurzeln besitzen, und im Grenzfalle erhalten wir die Abschätzungen von J. Favard (2), welche M. Krein und Autor unabhängig von ihm gefunden haben.

Institut für Mathematik und Mechanik
Charkow Universität.

Eingegangen
am 14. XI. 1937.

ZITIERTE LITERATUR

[1] Nörlund, Differenzenrechnung, S. 35. [2] J. Favard, C. R. (1936); N. Achyeser et M. Krein, C. R. Acad. Sci. URSS, XV, № 3 (1937).

* Wir wollen bemerken, dass aus dem hier Bewiesenen folgt, dass für ein nichtganzes λ $(0 < \lambda < n)$ die Gleichung

$$\cos \lambda u + a_0 + a_1 \cos u + \ldots + a_{n-1} \cos (n-1)u = 0$$

im Intervalle $0 < u < \pi$ immer nicht mehr als n Nullstellen haben kann.

Comptes Rendus (Doklady) de l'Académie des Sciences de l'URSS
1937. Volume XVII, № 9

MATHÉMATIQUES

SUR L'EXISTENCE D'UN INVARIANT INTÉGRAL

Par A. MARKOFF

(Présenté par S. N. Bernstein de l'Académie, le 14. XI. 1937)

1. Soit E un ensemble quelconque, $\mathfrak{M}$ une famille additive (1) de sous-ensembles de E. Nous appellerons les éléments de $\mathfrak{M}$ ensembles mesurables.

Soit μ une fonction réelle finie non négative et complètement additive (1) d'un ensemble mesurable. Le nombre $\mu(A)$ s'appellera mesure de l'ensemble A. À la mesure μ correspondent des notions d'une fonction sommable et de l'intégrale indéfinie (1).

Soit Γ un ensemble abélien (2) de transformations de l'ensemble E en lui-même. Nous supposerons que les relations $A \in \mathfrak{M}$, $\varphi \in \Gamma$ entraînent $\varphi^{-1}(A) \in \mathfrak{M}$. L'intégrale indéfinie d'une fonction f non négative, sommable, d'un élément de E sera dite invariant intégral par rapport à Γ lorsque

$$(\mu)\int_A f(x)\,dx = (\mu)\int_{\varphi^{-1}(A)} f(x)\,dx \qquad (A \in \mathfrak{M},\ \varphi \in \Gamma), \tag{1}$$

$$\mu\Big(\mathop{\mathcal{E}}_{x}\big(f(x)=0\big)\Big)=0. \tag{2}$$

2. Cette note est consacrée au problème d'existence d'un invariant intégral. Un cas particulier de ce problème, notamment le cas d'un ensemble Γ consistant d'une seule transformation biunivoque, a été considéré par E. Hopf, qui a établi une condition nécessaire et suffisante assez compliquée pour l'existence d'un invariant intégral (3). Plus antérieurement G. D. Birkhoff et P. A. Smith, en considérant ce même cas, ont obtenu une condition suffisante très simple (4). Cependant la condition de Birkhoff et Smith n'est pas nécessaire.

Le but de cette note est d'établir une condition simple, nécessaire et suffisante pour l'existence d'un invariant intégral.

3. L'ensemble consistant de la transformation identique de l'ensemble E et de tous les produits finis des éléments de Γ sera désigné par $[\Gamma]$.

Théorème. *Pour qu'il existe un invariant intégral par rapport à* Γ, *il faut et il suffit que quel que soit* $\varepsilon > 0$ *on peut prendre un* $\delta > 0$ *tel que les relations* $\mu(A) < \delta$, $\varphi \in [\Gamma]$ *entraînent* $\mu\big(\varphi^{-1}(A)\big) < \varepsilon$ *et que les relations* $B \in \mathfrak{M}$, $\psi \in [\Gamma]$, $\mu\big(\psi^{-1}(B)\big) < \delta$ *entraînent* $\mu(B) < \varepsilon$.

Voici une esquisse de la démonstration.

4. Nécessité. Admettons qu'il existe un invariant intégral

$$\mu^*(A) = (\mu) \int_A f(x)\,dx \qquad (A \in \mathfrak{M}); \tag{3}$$

alors μ^* est une fonction non négative et complètement additive d'un ensemble mesurable. Il résulte de (2) que l'égalité $\mu^*(A) = 0$ est équivalente à l'égalité $\mu(A) = 0$. On en conclut d'après le théorème de Radon-Nikodym (1) qu'on peut représenter la fonction μ sous la forme

$$\mu(A) = (\mu^*) \int_A f^*(x)\,dx \qquad (A \in \mathfrak{M}), \tag{4}$$

où f^* est une fonction non négative sommable (μ^*).

Soit maintenant ε un nombre positif arbitraire. Il résulte de la formule (4) qu'il existe un $\eta > 0$ tel qu'on a $\mu(A) < \varepsilon$ dès que $\mu^*(A) < \eta$. D'après (3) il existe un $\delta > 0$ tel qu'on a $\mu^*(A) < \eta$ dès que $\mu(A) < \delta$. En appliquant la condition (1), qui peut être écrite sous la forme

$$\mu^*(A) = \mu^*\left(\varphi^{-1}(A)\right) \qquad (A \in \mathfrak{M},\ \varphi \in [\Gamma]), \tag{5}$$

on montre aussitôt que δ jouit des propriétés requises.

5. Suffisance. En fixant $\varphi \in [\Gamma]$, nous pouvons considerer le produit $\varphi\chi$ comme fonction de la variable $\chi \in [\Gamma]$. Nous désignerons cette fonction par $\tilde{\varphi}$ en posant

$$\tilde{\varphi}(\chi) = \varphi\chi \qquad (\varphi \in [\Gamma],\ \chi \in [\Gamma]).$$

Nous désignerons par $\tilde{\Gamma}$ l'ensemble de tous les $\tilde{\varphi}$ correspondants aux éléments φ de $[\Gamma]$. On voit aussitôt que $\tilde{\Gamma}$ est un ensemble abélien de transformations de l'ensemble $[\Gamma]$ en lui-même.

On peut donc appliquer à $\tilde{\Gamma}$ le théorème d'existence des moyennes invariantes, contenu dans ma note précédente (2). Il existe d'après ce théorème une fonctionnelle M, définie dans la classe des fonctions réelles bornées d'un élément de $[\Gamma]$, et satisfaisante aux conditions:

$$M(g+h) = M(g) + M(h), \tag{6}$$

$$M(g) \geqslant 0, \text{ si } g(\varphi) \geqslant 0 \text{ pour chaque } \varphi \in [\Gamma], \tag{7}$$

$$M(g) = 1, \text{ si } g(\varphi) = 1 \text{ pour chaque } \varphi \in (\Gamma), \tag{8}$$

$$M(g\tilde{\varphi}) = M(g) \qquad (\varphi \in [\Gamma]), \tag{9}$$

où g et h sont des fonctions réelles et bornées, arbitraires, définies dans $[\Gamma]$.

Posons

$$g_A(\varphi) = \mu\left(\varphi^{-1}(A)\right) \qquad (A \in \mathfrak{M},\ \varphi \in [\Gamma]),$$

en définissant ainsi une fonction réelle, non négative g_A d'un élément de $[\Gamma]$. Cette fonction, dépendante d'un ensemble mesurable A, est bornée puisqu'on a $E \in \mathfrak{M}$ et $\mu(B) \leqslant \mu(E)$ pour chaque $B \in \mathfrak{M}$. La fonction g_A est donc admissible comme argument de la fonctionnelle M. Posons

$$\mu^*(A) = M(g_A) \qquad (A \in \mathfrak{M}), \tag{10}$$

en définissant par cette égalité une fonction réelle μ^* d'un ensemble mesurable.

Il résulte de (6), (7), (8) qu'on a

$$\inf g \leqslant M(g) \leqslant \sup g. \tag{11}$$

Les fonctions g_A étant non négatives, on en conclut que μ^* est une fonction non négative d'un ensemble.

Soit maintenant $A \in \mathfrak{M}$, $B \in \mathfrak{M}$, $A \frown B = \Lambda$. Quel que soit $\varphi \in [\Gamma]$ on a $\varphi^{-1}(A) \frown \varphi^{-1}(B) = \Lambda$, d'où $\mu\left(\varphi^{-1}(A \smile B)\right) = \mu\left(\varphi^{-1}(A) \smile \varphi^{-1}(B)\right) = \mu\left(\varphi^{-1}(A)\right) + \mu\left(\varphi^{-1}(B)\right)$, c'est-à-dire $g_{A \smile B} = g_A + g_B$. En vertu de (6) et (10) on obtient:

$$\mu^*(A \smile B) = \mu^*(A) + \mu^*(B) \qquad (A \in \mathfrak{M},\ B \in \mathfrak{M},\ A \frown B = \Lambda). \quad (12)$$

Supposons maintenant que la condition, dont la suffisance nous voulons démontrer, est satisfaite et montrons qu' alors μ^* est un invariant intégral par rapport à Γ. Fixons $\varepsilon > 0$ et prenons $\delta > 0$ de manière que l'inégalité $\mu\left(\varphi^{-1}(A)\right) < \varepsilon$ soit une conséquence des relations $\varphi \in [\Gamma]$, $\mu(A) < \delta$. Pour chaque ensemble A satisfaisant à la dernière inégalité nous avons sup $g_A \leqslant \varepsilon$, d'ou en vertu de (10) et (11) $\mu^*(A) \leqslant \varepsilon$. Il existe donc pour chaque $\varepsilon > 0$ un $\delta > 0$ tel que $\mu(A) < \delta$ entraîne $\mu^*(A) \leqslant \varepsilon$. Il en résulte que $\mu(A) = 0$ entraîne $\mu^*(A) = 0$.

Considérons une suite arbitraire $\{A_i\}_{i=1}^{\infty}$ d'ensembles disjoints mesurables. En vertu de l'additivité complète de μ nous avons

$$\lim_{n \to \infty} \mu\left(\bigsmile_{i=n+1}^{\infty} A_i\right) = 0.$$

Vu la propriété de μ^* qui vient d'être démontrée, on en déduit que

$$\lim_{n \to \infty} \mu^*\left(\bigsmile_{i=n+1}^{\infty} A_i\right) = 0,$$

d'où selon (12)

$$\mu^*\left(\bigsmile_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{n} \mu^*(A_i) + \mu^*\left(\bigsmile_{i=n+1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu^*(A_i).$$

La fonction μ^* est donc complètement additive. En vertu des propriétés établies de μ^* on conclut d'après le théorème de Radon-Nikodym [1] qu'il existe une fonction f non négative, sommable, satisfaisant à (3).

Fixons $\varepsilon > 0$ et prenons $\delta > 0$ de manière que l'inégalité $\mu(A) < \varepsilon$ soit une conséquence des relations $A \in \mathfrak{M}$, $\varphi \in [\Gamma]$, $\mu(\varphi^{-1}(A)) < \delta$. Pour chaque ensemble A tel que $\mu(A) \geqslant \varepsilon$ nous avons alors $\inf g_A \geqslant \delta$, d'où d'après (10) et (11) $\mu^*(A) \geqslant \delta$. Il existe, donc, pour chaque $\varepsilon > 0$ un $\delta > 0$ tel que $\mu^*(A) < \delta$ entraîne $\mu(A) < \varepsilon$. Il en résulte que $\mu^*(A) = 0$ entraîne $\mu(A) = 0$. En vertu de (3) cela donne l'égalité (2). Reste à montrer que la condition (1) est aussi satisfaite.

Pour $A \in \mathfrak{M}$, $\varphi \in [\Gamma]$, $\chi \in [\Gamma]$ nous avons $g_{\varphi^{-1}(A)}(\chi) = \mu(\chi^{-1}(\varphi^{-1}(A))) = \mu((\varphi\chi)^{-1}(A)) = g_A(\tilde{\varphi}(\chi)) = (g_A\tilde{\varphi})(\chi)$. Donc

$$g_{\varphi^{-1}(A)} = g_A\tilde{\varphi} \qquad (A \in \mathfrak{M},\ \varphi \in [\Gamma]),$$

et d'après (9) et (10) cela donne l'égalité (5). En vertu de (3) et (5) la condition (1) est satisfaite; c. q. f. d.

6. Considérons le cas particulièrement important pour les applications, dans lequel Γ est un groupe abélien de transformations biunivoques de l'ensemble E en lui-même. Dans ce cas notre condition d'existence d'un invariant intégral peut être encore simplifié. Comme corollaire du théorème démontré nous obtenons la proposition:

Si Γ est un groupe abélien de transformations biunivoques de l'ensemble E en lui-même, alors la condition suivante est nécessaire et suffi-

sante pour l'existence d'un invariant intégral par rapport à Γ: *quel que soit* $\varepsilon > 0$ *il existe un* $\delta > 0$ *tel que les relations* $\mu(A) < \delta$, $\varphi \in \Gamma$ *entraînent* $\mu(\varphi(A)) < \varepsilon$.

Institut de mathématiques
et de mécanique.
Université de Léningrad.

Manuscrit reçu
le 14. XI. 1937.

LITTÉRATURE CITÉE

[1] S. Saks, Théorie de l'intégrale (1933), Annexe. [2] C. R. Acad. Sci. URSS, I (X), 311—313 (1936). [3] E. Hopf, Trans. Amer. Math. Soc., **34**, 373—393 (1932). [4] G. D. Birkhoff and P. A. Smith, Journal de Math., 9 série, **7**, 345—379 (1928).

Comptes Rendus (Doklady) de l'Académie des Sciences des l'URSS
1937. Volume XVII, № 9

PHYSICS

NATURE OF ELEMENTARY SOURCES OF LIGHT AND INTERFERENCE PHENOMENA

By S. I. WAWILOW, Member of the Academy

§ 1. It is tacitly assumed in the interference theory and its various applications that the distinction between sources as to interference is determined by their spectra only. This is true for the Newtonian case of interference, when coherent beams are obtained by splitting up of a primary beam on reflection and refraction. In the case of Fresnel interference, however, when individual rays issuing from the same luminous point are brought together, the intensity distribution in the interference pattern, as well as the polarization of the fringes should depend, generally speaking, on the nature of elementary source, namely on the manner in which its radiation intensity is distributed along various directions in space. The usual idealization corresponding to an isotropic luminous point is faulty, as it corresponds to no real case of elementary sources.

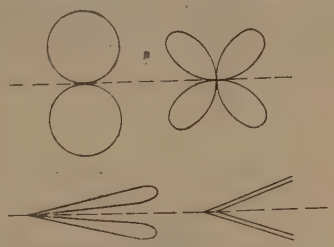


Fig. 1

All types of radiation may be grouped under four fundamental headings: 1) dipole radiation: 2) quadrupole radiation: 3) «Bremsestrahlung»; 4) radiation of electrons moving in a medium with velocity in excess of phase velocity of light(1). Octopole radiation as well as that of higher orders are omitted from this list, as having no practical importance. In all these four cases the elementary radiation (classically treated) is anisotropic.

The electric vector of a linear dipole varies as $\sin \alpha$ (α being the angle between the dipole axis and the line of sight); for a linear quadrupole the space distribution is represented by the function $\sin 2\alpha$ (2); in the case of «Bremsestrahlung» of an electron at speeds considerably below the velocity of light the electric vector is proportional to $\sin \alpha$ just as for a dipole; at speeds approaching the velocity of light the amplitude of radiation is proportional to $\frac{\sin \alpha}{1-\beta \cos \alpha}$. Finally in the case of uniform motion of an electron in a medium with speed exceeding the phase velocity of light the radiation must be concentrated within a narrow conical sheet, the generatrix of the cone forming an angle θ with the direction of electron motion, where $\cos \theta = \frac{1}{\beta n}$ (n being the refractive index of the medium). Plane sections of the space tadiation diagrams are given for all cases in Fig. 1. It is immediately evident from these diagrams that the amplitudes of two coherent rays issuing from the ele-

mentary source and inclosing a certain angle φ will be generally different, and thus the visibility of the interference pattern will depend on φ. In the quadrupole case the phases of interfering coherent rays will likewise differ. Moreover, as the coherent rays have to be deflected in the case of interference with aid of mirrors or prisms, here will be a relative change in the phases of the horizontal and vertical component of the rays brought together (3), and, therefore, the interference must be attended with polarization phenomena, depending likewise on the angle φ between the interfering rays. Thus the character of the interference pattern and its polarization must vary for different sources, notwithstanding the isotropic and unpolarized character of the total radiation.

§ 2. The case of dipole radiation has been worked out some years ago from this standpoint by the present writer and E. M. Brumberg (3). Let us denote by I and i the maximum resp. the minimum intensity in the interference fields; then the visibility of interference or interferential power v will be defined by the expression:

$$v = \frac{I - i}{I + i}. \qquad (1)$$

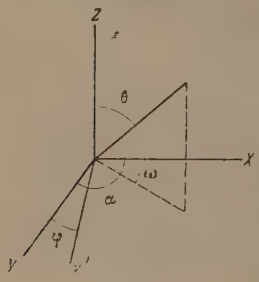


Fig. 2

A light source with dimensions less than the wave length (this being the necessary condition for realization of Fresnels experiment at large values of angle φ), consisting of irregularly distributed elliptical dipoles, was considered and it was shown that, independently of the eccentricity of the ellipses:

$$v = \frac{1 + \cos\varphi}{2}, \qquad (2)$$

φ being the angle between interfering rays. The interference field is found to be polarized. The degree of polarization at maximum of bright fringes is

$$P_{\max} = \frac{1 - \cos\varphi}{1 + \cos\varphi}, \qquad (3)$$

while at minimum of dark fringes it equals

$$P_{\min} = -1. \qquad (3')$$

The theoretical deduction has been confirmed by interference experiment with angle between rays close to 180°. In this case we have from Formula (2) $v = 0$, i. e. there is no interference, but the light field should be alternatively polarized [see Formulae (3) and (3')]. On observation through Nicol prism, the interference fringes are seen effectively to appear. Thus the dipole character of radiation has been proved by direct interference experiment.

§ 3. Let us examine a light source consisting of irregularly distributed linear quadrupoles. As mentioned above, the linear quadrupoles are characterized by electric vector of radiation being proportional to $\sin 2\alpha$ (α being the angle between the quadrupole axis and line of sight). Let a quadrupole with an axis whose direction is characterized by latitude θ and longitude ω be placed at the origin of a rectangular coordinate system (X, Y, Z), and let the Y axis coincide with the line of sight (Fig. 2). The value of electric vector of the radiation for the observer will be

$$E = \sin 2\alpha = 2\sqrt{1 - \sin^2\theta \sin^2\omega} \cdot \sin\theta \sin\omega.$$

Owing to transversality of light vibrations, this vector lies in the X, Z plane, its X and Z components being expressed as follows

$$X = 2\sin^2\theta \sin\omega \cos\omega; \quad Z = 2\sin\theta \cos\theta \cos\omega. \qquad (4)$$

Let us choose another direction of the line of sight Y, forming an angle φ with the original direction (Fig. 2). We have correspondingly

$$X' = 2\sin^2\theta\sin(\omega-\varphi)\cos(\omega-\varphi);\quad Z' = 2\sin\theta\cos\theta\sin(\omega-\varphi). \qquad (5)$$

By integration along the sphere we obtain the expression for the visibility of interference of the totality of irregularly distributed linear quadrupoles

$$\varphi = \frac{\int_0^{2\pi} d\omega \int_0^{\frac{\pi}{2}} [2(XX' + ZZ')]\sin\theta\, d\theta}{\int_0^{2\pi} d\omega \int_0^{\frac{\pi}{2}} [X^2 + X'^2 + Z^2 + Z'^2]\sin\theta\, d\theta}. \qquad (6)$$

After performing the integrations we obtain

$$\varphi = \frac{\cos 2\varphi + \cos\varphi}{2} = \frac{1+\cos\varphi}{2} - \sin^2\varphi. \qquad (7)$$

Curve I on Fig. 3 shows the variation of visibility of the interference pattern for a dipole according to formula (2), while curve II represents the corresponding value for a quadrupole according to formula (7). The paradoxical shape of this latter curve may be fully explained qualitatively by the radiation diagram of a linear quadrupole (Fig. 1), if the change of sign of the phase in adjacent strips of plane diagram be taken into account. Negative values of visibility correspond to the transition from a bright fringe to a dark one. The values of X, X', Z, Z' components permit to compute the degree of polarization at the maximum of a «bright» fringe and at the minimum of a «dark» one, in the same manner as it was made previously for a dipole:

$$P_{\max} = \frac{\cos\varphi - \cos 2\varphi}{2 + \cos\varphi + \cos 2\varphi}, \qquad (8)$$

$$P_{\min} = -\frac{\cos\varphi - \cos 2\varphi}{2 - \cos\varphi - \cos 2\varphi}. \qquad (8')$$

The full lines I and II on Fig. 4 represent $P_{\max}$ and $P_{\min}$, for a dipole, according to formulae (3) and (3′), while the dashed lines III and IV depict the formulae (8) and (8′) for a quadrupole.

§ 4. It has been mentioned already in § 1 that the «Bremsestrahlung» of slow electrons corresponds to the dipole case, and therefore interference phenomena cannot be used to discriminate it from other types of dipole radiation. But at electron speeds approximating the velocity of light the space distribution of radiation is totally changed (Fig. 1) and becomes similar to the conical radiation of an electron moving in a medium with speed exceeding the phase velocity of light. The latter case is especially interesting for it is open to experimental study. A strictly directive radiation of such a type might be realized by traversing a medium with a parallel electron beam of definite speed. In actual conditions (especially in the case of γ-ray excitation) electrons of different directions and speeds are present, and anisotropy of radiation is less pronounced. When the source of radioactive radiation is placed within the medium, there will be, generally, no directivity and radiation must be perfectly isotropic and unpolarized. However, in this case the study of interference features of radiation must disclose in an especially sharp manner the anisotropy of the elementary source.

Let us examine a volume element with dimensions less than a light wave

length its luminosity being produced by electrons moving at random within it with equal speeds exceeding the phase velocity of light. Owing to one-sidedness of elementary radiation and of its being confined to a thin conical sheet there may be evidently no coherent ryas enclosing an angle in excess of 2θ (θ being determined by the formula $\cos\theta = \frac{1}{\beta n}$), i. e. interference must be unquestionably totally absent if $\varphi > 2\theta$. If the conical sheet to which the elementary radiation is confined, is very thin, then at lesser angles the interference power must be practically absent likewise, and only at $\varphi = 2\theta$ some interference may be noticed, that will be excessively weakened owing to presence of incoherent rays from the remaining electrons. When electrons of different velocities are present, the phenomena become more complicated, and feeble interferences should be observable within a larger angular interval. Thus the luminosity discovered by P. A. Cherenkov should possess a remarkable feature of diminished interference power, provided the conical sheet of elementary radiation be very thin.

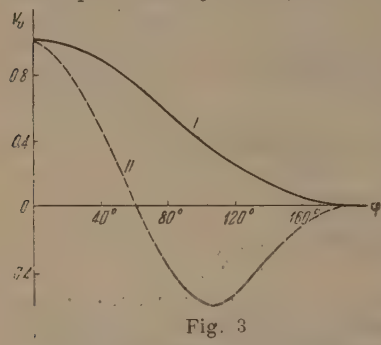


Fig. 3

The cases examined that practically exhaust the different types of sources, show that the study of interference power is a peculiar method permitting to discern at least three types of radiation. The method has been experimentally proved for the dipole case ([3]), the study of quadrupole radiation being equally possible while the luminosity discovered by P. Cherenkov is at present under investigation.

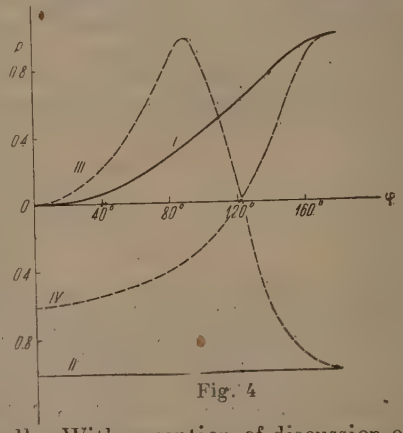


Fig. 4

N o t e. After completion of the present paper, a contribution by Halpern and Dorfmann ([4]) to the same problem became known to us, which has appeared in November 1937. The authors have not been acquainted, evidently, with our 1932 paper ([3]), in which the dipole radiation was analysed theoretically as well as experimentally. With exception of discussion of quadrupole and octupole cases, the work of Halpern and Dorfmann does repeat the theoretical part of the just mentioned paper. The conclusions of the present paper about quadrupole radiation overlap partially with those of Halpern and Dorfmann, although they are obtained in a different way. The problems of «Bremsestrahlung» are not discussed by Halpern and Dorfmann.

Optical Institute.
Leningrad.

Received
10. XII. 1937.

REFERENCES

[2] P. A. Cherenkov, C. R. Acad. Sci. URSS, VIII, 451 (1934); XII (3) 413 (1936); XIV, 102 (1937); XIV. 105 (1937); Phys. Rev., **52**, 378 (1937); S. I. Wawilow, C. R. Acad. Sci URSS, VIII, 455 (1934); I. M. Frank a. I. E. Tamm, C. R. Acad. Sci. URSS, XIV, 109 (1937). [2] A. Rabinowicz u. I. Blaton, Erg. der exakten Naturwissenschaften, 11 (1932). [3] S. I. Wawilow u. E. M. Brumberg, Phys. ZS. d. Sov. Union, **3**, 103 (1932); S. I. Wawilow, Bull. Acad. Sci. URSS, 1451 (1932). [4] O. Halpern a. F. Dorfmann, Phys. R. **52**, 937 (1937).

Comptes Rendus (Doklady) de l'Académie des Sciences de l'URSS
1937. Volume XVII, № 9

PHYSICS

THE LOW ENERGY REGION OF THE β-SPECTRUM OF RaE

By A. I. ALICHANIAN and A. S. ZAVELSKIJ

(Communicated by A. F. Joffe, Member of the Academy, 25. XI. 1937)

It was shown by A. Alichanian, A. Alichanow and B. Dželepow (1) that the character of the β-spectrum of RaE in the low energy region is very different from that of light elements. The authors attributed the presence of a large number of slow electrons in the spectrum of RaE to the effect of the Coulomb field of the disintegrating nucleus. The effect of this field may be taken into quantitative account in the theories of Fermi and Uhlenbeck-Konopinski (2) by computing a certain function

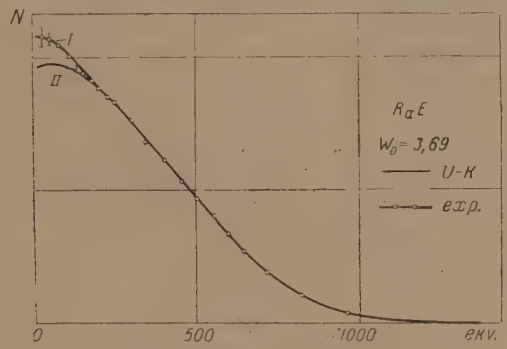


Fig. 1.

$f(z, w)$. A knowledge of the form of the distribution curve in the region of the lowest energies (10—100 ekV) is of considerable interest since it enables important factors controlling the form of the β-spectrum to be determined. In the same paper it was shown that the distribution curve for RaE does not start from the zero coordinate but intersects the ordinate axis. In these investigations, however it was necessary to introduce a correction, which for 30—50 kV energy was considerable, owing to a distortion in the region 30—100 kV caused by electron absorption in the window of the counter. The necessity for this correction naturally reduced the accuracy of the results.

In the present investigation magnetic focussing and registration by means of a counter were also employed, the window being covered by a very thin film of acetylcellulose ($\sim 1\,\mu$). This enabled electrons of only

7—8 ekV to pass into the counter whilst the correction for absorption in the film was only 1% for 100 kV electrons and 4% for 40 kV. The source was a narrow strip of aluminium 0.5 μ thick on which RaE was deposited by electrolysis. The distribution curve which starts form 25 ekV is given in Fig. 1 and shows that there is a steady increase in the number of electrons with decreasing energy. In Fig. 2 the $u - k$ plot is given which shows that the Uhlenbeck-Konopinski theory is in excellent agreement with the experimental data from the very smallest energies. Since the Uhlenbeck-Konopinski plot is not very sensitive to a change in the ordinate of the spectrum, it is of interest to compare directly the theoretical and experimental distribution curves. Using the spectral limit obtained by extrapolation of the Uhlenbeck-Konopinski plot we have constructed the theoretical curve reproduced in Fig. 1, curve *II*. Clearly the two curves are in excellent agreement over practically the whole range. This indicates that not only the statistical distribution determined by the factor $w \cdot (w-1)^{1/2} \cdot (w_0 - w)^4$ but also the function $f(z, w)$ which accounts for the influence of the Coulomb field are given correctly and with high accuracy by the theory. A certain divergence occurs for electron energies below 100 kV where the experimental spectrum is higher than the theoretical and reaches 10% for an energy of 50 kV. This divergence scarcely can be accounted for by experimental errors since these did not exceed 3—5% in this region. It is possible that the difference between experiment and theory may be due to the fact that the β-decay of RaE belongs to the class of forbidden transitions accompanied by a change of the angular moment of the nucleus ($\Delta i \neq 0$). The theoretical spectra *II* given in Fig. 1 referes however to an allowed transition ($\Delta i = 0$). The large difference between the Uhlenbeck-Konopinski and the experimental spectra in the vicinity of the limit will not be discussed here.

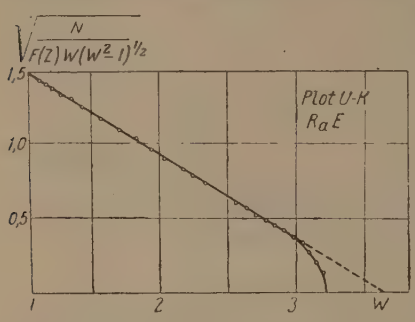


Fig. 2.

Physico-Technical Institute.
Leningrad.

Received
25. XI. 1937.

REFERENCES

[1] A. I. Alichanian, A. I. Alichanow a. B. S. Dželepow, Phys. ZS. d. Sowietunion, II, H. 2 (1937). [2] Konopinski a. Uhlenbeck, Phys. Rev., 48, 7 (1935).

Comptes Rendus (Doklady) de l'Académie des Sciences de l'URSS
1937. Volume XVII, № 9

PHYSICS

THE LOW ENERGY REGION OF THE β-SPECTRA OF ThC'' and ThB

By A. I. ALICHANIAN and A. S. ZAVELSKIJ

(Communicated by A. F. Joffe, Member of the Academy, 25. XI. 1937)

In the previous note we reported * the results of measurements of the β-spectrum of RaE in the region of low energies. They emphasized the necessity of studying the β-spectra of heavy radioactive elements having different maximum energies. The β-spectra of ThC'' and ThB have therefore been investigated by means of the magnetic focussing method.

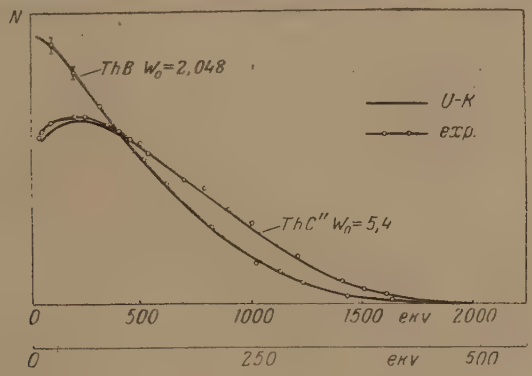


Fig. 1.

The atoms of ThC'' were obtained from an active thorium deposit by the metod of radioactive recoil. Since the half-period of ThC'' is only 3.1 minutes it was necessary to place new samples of ThC'' atoms at the focus of the apparatus at definite time intervals without disturbing the vacuum. In order to do this, a ground joint was provided in the wall of the vessel and the active thorium deposit could be introduced by a simple rotation. By means of another ground joint an aluminium foil 0.5 μ thick could be brought almost into contact with the deposit and after saturation with ThC'' atoms the foil could be rotated into the focal position. The spectrum of ThC'' is given in Fig. 1. The total number of electrons registered in this case exceeded 100 000. In addition to this spectrum, that of the active thorium deposit [Th(B+C+C'')] was also

* See precedent note.

taken. Here the main part of the spectrum representing the low energy electrons is due to the disintegration of ThB. From the ThC'' spectrum and that of ThC constructed by analogy, it is possible to obtain the ThB spectrum by subtracting the β-spectrum of Th(C+C'') from the total. The result is very little affected by the method adopted for the construction of the ThC spectrum since in the low energy region the number of electrons entering the spectrum of Th(C+C''), is only $\sim 20\%$ of the number in the spectrum of Th(B+C+C''). The spectrum of ThB is also given in Fig. 1. The Uhlenbeck-Konopinski plot has been calculated for both the ThC'' and ThB spectra (Fig. 2). The straight lines obtained indicate excellent agreement between the experimental spectra and the Uhlenbeck-Konopinski distribution. Only in the region of the upper limit is there some serious departure. Using the limits of spectra obtained by extrapolation of the Uhlenbeck-Konopinski plot we have constructed the theoretical spectra of ThC'' and ThB indicated on Fig. 1. This method of comparing the theoretical and experimental curves is considerably more sensitive than the method of plot and shows that the theoretical form of the β-spectrum is correct to a high degree of accuracy. On the basis of the analysis of the spectra of the light elements, studied with the same degree of care, and of the heavy elements given above, it may be affirmed that the main factors determining the form of the β-spectra, namely the statistical distribution and the function $f(z, w)$ as given by the theory are in excellent agreement with experiment. It should be mentioned that in contrast to RaE the agreement holds down to the smallest energies.

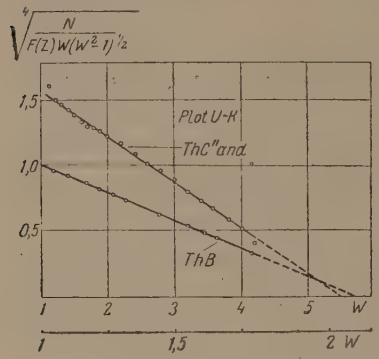


Fig. 2.

This supports the proposition that the disagreement found in the case of the spectrum of RaE may be due to the fact that the β-disintegration of RaE belongs to the category of forbidden transitions. Thus it is well known that the transitions in ThC'' and ThB are allowed ($\Delta i = 0$) and the theoretical spectra were calculated for this type of transition. It is of interest to note that the extrapolated spectral limit of ThB (540 kV) considerably exceeds (by about 35%) the experimental value (410 kV) whilst the difference in the case of other spectra is only 10—20%. It is possible that other factors which are not taken account of by the theory (e. g. the finite mass of the neutrino) and which give rise to a divergence between the theoretical and experimental spectral limits are of greater influence when the maximum disintegration energy is relatively low.

Physico-Technical Institute.
Leningrad.

Received
25. XI. 1937.

Comptes Rendus (Doklady) de l'Académie des Sciences de l'URSS
1937. Volume XVII, № 9

PHYSIQUE MATHÉMATIQUE

SUR LES VIBRATIONS TAUTOCHRONES DANS LES SYSTÈMES CONSERVATIFS ET NON CONSERVATIFS

Par I. KOUKLES et N. PISKOUNOV

(Présenté par I. M. Vinogradow, de l'Académie, le 14. X. 1937)

Considérons une équation des vibration à un degré de liberté:

$$\frac{d^2x}{dt^2} + F\left(x, \frac{dx}{dt}\right) = 0. \tag{1}$$

En posant $\frac{dx}{dt} = v$ nous donnons à cette équation la forme suivante:

$$\frac{dv}{dx} + \frac{F(x, v)}{v} = 0; \quad \frac{dx}{dt} = v. \tag{2}$$

Supposons maintenant que l'origine des coordonnées est un point singulier qui est un foyer ou bien un centre. Considérons la période des vibrations définies par le système (2), c'est-à-dire le temps pendant lequel un point variable fait dans le plan (x, v) un tour complet autour de l'origine des coordonnées. Cette période T dépend, en général, des conditions initiales c'est-à-dire de l'amplitude a ou de la vitesse initiale (α). Nous appellerons «vibrations tautochrones» celles qui ont une période ne dépendant pas des conditions initiales.

Nous allons distinguer trois sortes de tautochronisme:

1) Le tautochronisme faible—c'est le cas où T ne dépend pas des conditions initiales.

2) Le tautochronisme fort—c'est le cas où le tautochronisme faible a lieu et d'ailleurs la période ne dépend pas de la position du point variable quand l'angle de rotation varie de $-\frac{\pi}{2}$ à $+\frac{\pi}{2}$.

3) Le tautochronisme parfait — c'est le cas où les vitesses angulaires de tous les points situés sur un même rayon vecteur sont égales.

Théorème 1. *Pour que le tautochronisme soit parfait il faut et il suffit que* $F(x, v) = \lambda x + kv$.

Il en résulte que le tautochronisme d'Abel est toujours parfait et, inversement, pour le cas d'un centre chaque tautochronisme parfait est le tautochronisme d'Abel *.

Théorème 2. *Dans le cas d'un foyer le tautochronisme est toujours parfait. Il en résulte que dans le cas d'un foyer non-linéaire le tautochronisme ne peut pas avoir lieu**.

Théorème 3. *Si dans l'équation (1) la fonction* $F(x, v)$ *et ses dérivées partielles par rapport à* x *et* v *s'annulent à l'origine des coordonnées, la période tend vers l'infini quand l'amplitude tend vers zéro.*

* Les théorèmes 1 et 2 sont démontrés en supposant que la fonction $F(x, v)$ est analytique au voisinage du point 0, 0.

Remarquons que dans les conditions de ce théorème l'origine peut être un foyer ou bien un centre. Mais il résulte du théorème que le tautochronisme est impossible dans ce cas.

Théorème 4. *Si* $F(x, v) = f(x) + \varphi(v)$ *où* $f(v)$ *et* $\varphi(v)$ *sont des polynômes, le tautochronisme fort n'a lieu que dans le cas où* $f(x) = \lambda x$ *et* $\varphi(v) = av$.

D'ailleurs on peut démontrer que si $F(x,v) = f(x) + \varphi(v)$, où f et φ sont des fonctions analytiques quelconques, l'existence du tautochronisme dans le demi-plan à droite (à gauche) de l'axe des ordonnées implique le tautochronisme à gauche (à droite) des cet axe, donc le tautochronisme est fort.

Ensuite, en considérant les vibrations dans les systèmes conservatifs, nous démontrons le théorème suivant:

Théorème 5. *Pour que la période des vibrations définies par l'équation* $\frac{d^2x}{dt^2} + f(x) = 0$ *soit égale à un certain nombre* C *qui ne dépend pas des conditions initiales il faut et il suffit que les conditions suivantes soient vérifiées:*

1) $f(x)$ *est continue et positive pour les valeurs de* x *positives et assez petites.*

2) $\liminf_{t \to 0} \left| \frac{f(t)}{t} \right| \neq 0.$

3) *On a l'inégalité:*

$$C \geqslant \limsup_{t \to 0} \frac{\Pi}{\sqrt{\frac{f(t)}{t}}}.$$

4) *On a l'égalité:*

$$f(-x) = \frac{d}{dx} \left\{ \left[\left(\int_0^x f(\alpha)\, d\alpha \right)^{-1} - \frac{C}{\pi} \sqrt{2x} \right]^{-1} \right\}. \tag{3}$$

(L'exposant $^{-1}$ désigne qu'on considère la fonction inverse. Cette fonction inverse existe parce que $\int_0^x f(\alpha)\, d\alpha$ est une fonction croissante pour les petites valeurs de x).

Ainsi, en vertu de ce théorème, nous pouvons prendre pour $f(x)$ une fonction quelconque vérifiant les conditions 1) et 2) pour les petites valeurs de $x > 0$ et la prolonger ensuite d'une manière unique [en vertu de 4)] pour les valeurs négatives de x en tenant seulement compte de ce que le tautochronisme ait lieu; alors la période des vibrations est égale à un nombre C quelconque vérifiant l'inégalité 3).

On démontre que la fonction $f(x)$ ainsi définie est discontinue au point $x = 0$ si $f(+0) \neq 0$, qu'elle est continue si $f(+0) = 0$ et qu'elle est continue, ainsi que sa dérivée, si $f(+0) = 0$, $f'(+0)$ existe et $C = \frac{2\pi}{\sqrt{f'(0)}}$. Si la condition 3) n'est pas vérifiée, le tautochronisme est impossible quelle que soit la manière de prolonger f.

Exemple 1.

Soit $f(x) = 1$ pour $x \geqslant 0$ et $C = \pi$. Pour $x \leqslant 0$ on trouve de la condition (4):

$$f(x) = x - \frac{1 \cdot 3}{2!} x^2 + \frac{1 \cdot 3 \cdot 5}{3!} x^3 - \frac{1 \cdot 3 \cdot 5 \cdot 7}{4!} x^4 + \ldots$$

Théorème 6. *Pour que les vibrations définies par l'équation* $\frac{d^2x}{dt^2} + f(x) = 0$ *soient tautochrones et la fonction* $f(x)$ *soit holomorphe au voisinage de l'origine des coordonnées il faut et il suffit que la fonction* $\Theta(z) = \left[\int_0^z f(\alpha)\, d\alpha\right]^{-1}$ *soit de la forme:*

$$x = \Theta(z) = \sqrt{kz} + P(z), \tag{4}$$

où k *est une constante arbitraire et* $P(z)$ *est une fonction holomorphe et d'ailleurs quelconque de la variable* z.

En d'autres termes, pour qu'un pendule soit tautochrone et analytique il faut et il suffit que la longueur de l'arc x de la courbe qu'il décrit soit exprimée en fonction de l'ordonnée z par l'égalité (4).

Exemple 2.

Soit $x = \Theta(z) = \sqrt{2z} + z$. On trouve que $T = 2\pi$ et que

$$f(x) = x - \frac{1\cdot 3}{2!}x^2 + \frac{1\cdot 3\cdot 5}{3!}x^3 - \frac{1\cdot 3\cdot 5\cdot 7}{4!} + x^4 + \ldots = 1 - (1+2x)^{-\frac{1}{2}}. \tag{5}$$

Cette fonction caractérise un pendule tautochrone non linéaire.

Nous voyons ainsi que pour un système conservatif les vibrations non linéaires ne peuvent posséder que le tautochronisme faible (d'après le théorème d'Abel). Nous verrons dans la suite que pour les systèmes non conservatifs les vibrations non linéaires peuvent posséder non seulement le tautochronisme faible mais encore le tautochronisme fort.

Théorème 7. *Quelles que soient la fonction* $f(x)$ $[f'(0) > 0]$ *et la fonction* $\psi(x)$, *il existe une infinité de fonctions* $\varphi(x)$ *vérifiant l'équation* $\frac{dv}{dx} = -\frac{f(x) + v^2\varphi(x) + v^4\psi(x)}{v}$ *et réalisant le tautochronisme faible tandis qu'il n'existe qu'une seule fonction* $\varphi(x)$ *vérifiant la même équation et réalisant le tautochronisme fort. Le même théorème subsiste pour la fonction* $f(x)$ *quand les fonctions* $\varphi(x)$ *et* $\psi(x)$ *sont données arbitrairement.*

Si $\psi(x) = 0$ *et* $\varphi(x)$ *est donnée, on peut trouver la fonction* $f(x)$ *réalisant le tautochronisme fort au moyen de quadratures, à savoir:*

$$f(x) = e^{-\int_0^x \varphi(\alpha)\, d\alpha} \int_0^x e^{\int_0^\alpha \varphi\beta\, d\beta}\, d\alpha. \tag{6}$$

Dans ce cas, si $f(x)$ est donnée, on trouve $\varphi(x)$ de la relation

$$\varphi(x) = -\frac{f'(x) - 1}{f(x)}. \tag{7}$$

Exemple 3.

Trouver la fonction $f(x)$ réalisant le tautochronisme fort dans les conditions: $\psi(x) = 0$; $\varphi(x) = \lambda = \text{const}$.

On obtient de la formule (6)

$$f(x) = \frac{1}{\lambda}(1 - e^{-\lambda x}).$$

Quand $\lambda \to 0$, on a $f(x) = x$, c'est-à-dire on est réduit au cas linéaire.

Théorème 8. *Quelle que soit la fonction* $f(x)$ $[f'(0) > 0]$ *il existe une infinité de fonctions* $\Phi(v)$ *vérifiant l'équation* $\frac{dv}{dx} = -\frac{f(x)\,\Phi(v)}{v}$ *et réalisant le tautochronisme faible, tandis qu'il n'existe qu'une seule fonction* $\Phi(v)$ *réalisant le tautochronisme fort.*

Le même théorème subsiste pour la fonction $f(x)$ si l'on suppose que $\Phi(v)$ est donnée arbitrairement.

Chaque fonction $F(x, v)$ réalisant le tautochonisme fort ($T = 1$) peut être trouvée de la manière suivante: on prend une fonction arbitraire $\Pi(x, a)$; on trouve la fonction N définie par la relation:

$$N(x, a) = \left[\frac{a\Pi(a, a)}{a\Pi'_x(x, a) - \Pi(0, a)} \right]^2. \tag{8}$$

On trouve ensuite

$$\frac{dN}{dx} = K(x, a) \tag{8'}$$

et en éliminant la constante a des formules (8) et (8') on trouve enfin $\frac{\partial N}{\partial x}$ comme fonction de N et x:

$$\frac{\partial N}{\partial x} = F(x, N). \tag{9}$$

La fonction $\frac{\partial N}{\partial x} = F(x, N)$ ainsi trouvée réalise le tautochronisme fort, c'est-à-dire les vibrations définies par l'équation $\frac{dv}{dx} = -\frac{F(x, v^2)}{v}$ possèdent le tautochronisme fort.

On a démontré que pour chaque fonction (Fx, v) réalisant le tautochronisme fort il existe une fonction $\Pi(x, a)$ qui permet d'obtenir $F(x, v)$ au moyen du procédé précédemment décrit.

Exemple 4. Soit $\Pi(x, a) = \Phi\left[\frac{f(x)}{f(a)}\right]$, où Φ et f sont des fonctions arbitraires. On trouve

$$N = F_1\left[\frac{f(x)}{f(a)}\right] \frac{f^2(a)}{f^2(x)}.$$

En éliminant $f(a)$, on obtient facilement:

$$F(x, v) = \frac{v^2}{\varphi(x)}\left[\omega\left(\frac{v}{\varphi(x)}\right) - \varphi'(x)\right], \tag{10}$$

où ω et φ sont des fonctions arbitraires. Si l'on pose $\omega(\alpha) = 1 + \frac{1}{\alpha^2}$, on obtiendra immédiatement:

$$F(x, v) = \varphi(x) - v^2\left[\frac{\varphi'(x) - 1}{\varphi(x)}\right], \tag{7'}$$

c'est-à-dire la formule (7) qui a été obtenue précédemment.

Lagrange, Darboux, Haton de la Goupillère et Hademard ont étudié le tautochronisme fort. En particulier, c'est Lagrange qui a obtenu la formule (8) que nous avons établi en considérant l'exemple 4. Lagrange croyait que sa formule est la plus générale, mais il resulte de la formule (6) et des théorèmes 7 et 8 qu'elle est loin d'être générale.

Darboux [1] avait analysé le tautochronisme fort dans le cas où $F(x, v) = f(x) + v^2\varphi(x)$ ce qui d'ailleurs a été fait précédemment par Euler [3].

Haton de la Goupillère [2] avait étudié le tautochronisme fort pour la fonction $F(x, v) = x + \varphi(v)$ et il a démontré que $\varphi(v) = av$, a étant une constante.

Hadamard [4] avait étudié le tautochronisme fort pour les systèmes conservatifs et il a donné une belle et très simple démonstration de ce que les vibrations linéaires sont les seules possibles dans ce cas. Mais

comme tous ses prédécesseurs Hadamard a supposé que la fonction $f(x)$ ne possède qu'un nombre fini de maxima et minima au voisinage de l'origine.

Dans la présente note nous avons facilement obtenu la formule

$$T = \frac{\Pi}{\sqrt{\frac{f(\eta)}{\eta}}},$$

où η est une variable qui tend vers zéro avec a. En supposant que $T = \text{const}$ nous obtenons immédiatement $f(\eta) = \lambda\eta$, ce qui nous permet de rejeter les restrictions d'Hadamard et des autres auteurs (Puisseux, etc.).

Il paraît que le tautochronisme faible n'a pas été étudié jusqu'à présent.

Il nous semble important de faire encore la remarque suivante: D. Andronoff [5] croit que les mouvements périodiques d'un système conservatif (non linéaire) sont toujours instables au sens de Liapounoff. Mais il suit des résultats précédemment exposés qu'il n'en est pas ainsi parce que les vibrations non linéaires peuvent posséder le tautochronisme faible dans le cas d'un système non conservatif; mais dans le cas du tautochronisme faible ces vibrations doivent être stables au sens de Liapounoff. Donc les systèmes conservatifs ne présentent aucune exception et se conduisent de même que les systèmes non conservatifs.

Institut mathématique V. A. Stekloff et
Institut sismologique.
Académie des Sciences de l'URSS.
Moscou.

Manuscrit reçu
le 21. XI. 1937.

LITTÉRATURE CITÉE

[1] Darboux, Note à la Mécanique de Despeyrons. [2] Haton de la Goupillière, Journ. de Liouville, XIII, 2-e série. [3] Julien, Les problèmes de la mécanique, 1. [4] Hadamard, Procès verbaux des séances de la soc. des Sciences phys. et nat. Bordeaux (1895). [5] D. Андронов, Доклад на Всесоюзной конференции по теории колебаний (1931).

Comptes Rendus (Doklady) de l'Académie des Sciences de l'URSS
1937. Volume XVII, № 9

CHIMIE

LES ACIDES HEXAHYDRO-BENZILE-BARBITURIQUES

Par M. M. KATZNELSON et D. A. BRODSKY

(Présenté par A. E. Favorskij, de l'Académie, le 3. X. 1937)

Vu la grande importance pharmacologique des dérivés barbituriques, ceux-ci, depuis longtemps, ont été l'objet d'une étude approfondie.

Après l'obtention du véronal par Emile Fischer(1), des centaines de dérivés barbituriques ont été décrits dans la littérature chimique. Jusqu'à ces derniers temps tout l'intérêt des savants a été concentré sur l'obtention de substitués d'acides barbituriques ayant 5-5 radicaux de la série grasse ou aromatique.

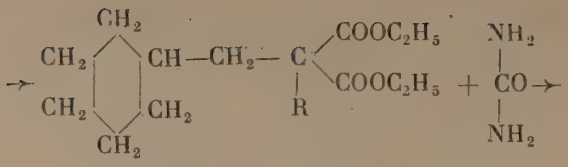
Un grand intérêt présentent les acides barbituriques substitués dans l'azote. Les acides barbituriques qui contiennent dans la position 5-5 des radicaux de la série alicyclique ont des propriétés pharmacologiques très intéressantes. Au nombre de ces composés appartient le phanadorme (acide cyclohexényleéthyle-barbiturique).

Au cas d'une combinaison du résidu cyclohéxényle avec le groupe CH_3, substitué dans l'azote, résulte un composé ayant un effet pharmacologique tout à fait original (évipan). Malgré le fait que l'introduction du résidu alicyclique dans l'acide barbiturique a comme suite l'obtention de nouvelles substances somnifères, dans la littérature on ne trouve que très peu de descriptions de ces composés. Notre travail a pour but de remédier à ce manque

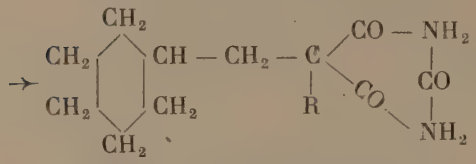
Nous nous sommes arrêtés à l'obtention des dérivés barbituriques qu dans la position 5-5 ont un résidu hexahydro-benzile. Une série analogue au groupe benzilique a été le plus profondement étudiée par A. Dox et Joder (2). Ces derniers ont obtenu les acides 5-5-méthyle-, éthyle-, propyle-, isopropyle-,isobutyle- et d'autres acides benzile-barbituriques. Un intérêt pharmacologique particulier présente l'acide 5-5-éthyle-benzile-barbiturique qui parmi les composés obtenus par Dox donne le plus grand effet pharmacologique. Néanmoins, au lieu de l'action antispasmatique(3), attendue du groupe benzilique, le composé mentionné ci-dessus a produit en même temps que l'action somnifère également des cas de tétanos.

Le problème que nous nous sommes posé était de trouver dans quelle mesure l'hydrogénisation du groupe benzilique de la série de composés étudiée par nous influencera l'augmentation de l'effet sommnifère et l'affaiblissement de l'effet de tétanos.

Nous avons obtenu une série d'acides hexahydro-benzile-barbituriques substitués (méthyle, éthyle, propyle, isobutyle, isoamyle) d'après le schéma suivant:



Éther alkyle hexahydro-benzile-malonique



Acide alkyle hexahydro-benzile-barbiturique

$$R = H, CH_3, C_2H_5, C_3H_7, C_4H_9, C_5H_{11}$$

Tous les acides que nous avons obtenus sont des composés qui se cristallisent parfaitement, se dissolvent facilement dans l'alcool, dans l'acétone, et ne se dissolvent pas dans le benzène, dans l'éther et dans l'eau. Les sels de sodium de ces acides sont obtenus par l'action de l'alcoolate de sodium sur les solutions alcooliques des acides barbituriques.

Les sels de sodium des acides mentionnés ci-dessus s'hydrolisent facilement si on les chauffe. Les composés que nous avons obtenus ont été pharmacologiquement étudiés par le docteur Merkulov à Léningrad dans le laboratoire pharmacologique du «VIEM» (Directeur prof. A. Kouznetzov); cette étude sera publiée.

Ces recherches pharmacologiques montrent que c'est le sel de sodium de l'acide hexahydro-benzile-éthyle-barbiturique qui en comparaison avec le médinale donne le plus grand effet somnifère, et ne produit pas de tétanos comme les acides 5-5-benzile-ethyle-barbituriques.

Dans la partie expérimentale, ci-dessous, nous allons faire la description du mode d'obtention des acides alkyle hexahydro-benzile-barbituriques et de quelques éthers alkyle hexahydro-benzile-maloniques.

Vu le grand intérêt pharmacologique de l'acide 5-5-hexahydro-benzile-éthyle-barbiturique on continue à étudier cette série et on poursuit des recherches dans cette direction.

PARTIE EXPÉRIMENTALE

Éther hexahydro-benzile-malonique

A 45 cm³ d'alcool absolu on ajoute partiellement 2.6 g de sodium métallique. Après dissolution complète du sodium dans l'alcool, on ajoute à la solution de l'éthylate de sodium 15 g d'éther malonique fraichement distillé. Au Na-éther malonique on ajoute au fur et à mesure 18 g de cyclohexyle-carbinole-bromide. On fait bouillir le mélange à reflux au bain marie pendant 5 à 6 heures. Ensuite on fait évaporer l'alcool aussi complètement que possible. Le résidu dans le ballon est dilué dans une petite quantité d'eau jusqu'à dissolution du bromure de sodium. La solution aqueuse est extraite par l'éther. La solution d'éther est séchée par le sulfate de sodium et filtrée; l'éther est évaporé et le résidu distillé dans le vide, obtenant une fraction bouillant à 122—124° sous pression de 4 mm; rendement 15 g (50% de la quantité théorique).

Acide hexahydro-benzile-barbiturique

Dans un tube soudé on chauffe pendant 6 heures à une température de 106 à 107° un mélange de 5 g d'éther hexahydro-benzile-malonique (fraichement distillé), de 2 g d'urée et de 20 cm³ d'une solution contenant 5% d'éthylate de sodium. Après refroidissement et ouverture du tube, le contenu est filtré. Le sel du sodium obtenu est dissout dans l'alcool, filtré, et l'alcool est complètement évaporé.

Le résidu ayant l'aspect d'une huile épaisse est lavé par l'éther absolu; ensuite il est dissout dans l'eau et la solution est filtrée. On agite le filtrat avec de l'éther pour éliminer les corps solubles dans l'eau. L'acide barbiturique se dépose lorsque la solution aqueuse du Na-sel devient acidulée donnant une réaction acide sur le congo. Le rendement du produit brut non récristallisé est de 2.47 g (60% de la quantité théorique). L'acide hexahydro-benzile-barbiturique se cristallise dans l'alcool méthylique présentant des lamelles bien formées, température de fusion 265—266° (non corrigée).

Analyse. Pesée 0.1352 g; 14.68 cm³ N (20°, 766.2 mm).
$C_{11}H_{16}N_2O_3$ Trouvé: N = 12.62%.
Calculé: N = 12.5%.

Éther hexahydro-benzile-méthyle-malonique

A 30 cm³ d'alcool absolu on ajoute partiellement 2.17 g de Na métallique, jusqu'à dissolution complète. A la solution de l'alcoolate de sodium on ajoute 16 g d'éther méthyle-malonique et 20 g de bromide cyclohexyle-carbinole. Le mélange est chauffé au bain marie pendant 7 heures. Après avoir fait évaporer l'alcool du mélange en réaction, on ajoute de l'eau au résidu du ballon jusqu'à dissolution complète du bromure du sodium. La solution aqueuse est extraite à plusieurs reprises par l'éther. Après l'avoir séchée au-dessus du sulfate de sodium, on la filtre et on évapore l'éther, on distille le résidu dans le vide obtenant une fraction bouillant à 135—136° sous pression de 1 mm; rendement 18 g (60% de la quantité théorique).

Acide hexahydro-benzile-méthyle-barbiturique

Dans un tube soudé on chauffe pendant 5 heures à une température de 104—105°, 10 g d'éther hexahydro-benzile-méthyle-malonique, 25 cm³ d'une solution contenant 5% d'éthylate de sodium, et 4 g d'urée.

Après l'ouverture du tube, son contenu est traité de façon employée pour l'obtention de l'acide hexahydro-benzile-barbiturique.

L'acide hexahydro-benzile-méthyle-barbiturique se cristallise dans l'alcool méthylique en forme d'aiguilles, température de fusion 223°.

Analyse. Pesée 0.0965 g; 9.81 cm³ N (19°, 766.7 mm).
$C_{12}H_{18}O_3N_2$ Trouvé: N = 11.73%.
Calculé: N = 11.71%.

Éther hexahydro-benzile-éthyle-malonique

Dans 40 cm³ d'alcool absolu on dissout 1.8 g de Na métallique. On ajoute à cette solution 15 g d'éther éthyle-malonique et 20 g de bromide cyclohexyle carbinole. On fait bouillir la solution pendant 8 heures. Le produit de la réaction est traité de la même manière que dans les cas décrits plus hauts. La solution aqueuse est extraite par l'éther et est séchée au-dessus du sulfate de sodium. Après évaporation de l'éther, le résidu est distillé dans le vide,

obtenant une fraction bouillant à 171°, sous pression de 20 mm; rendement 15.5 g (50% de la quantité théorique).

Acide hexahydro-benzile-éthyle-barbiturique

Dans un tube soudu on chauffe à une température de 110—115° pendant 5 heures 10 g d'éther hexahydro-benzile-éthyle-malonique (fraichement distillé, 3.4 g d'urée et 30 cm³ d'une solution contenant 4% d'éthylate de sodium. A la fin de la réaction on traite le contenu du tube de la manière que nous avons employée pour obtenir les acides barbituriques, décrits ci-dessus.

Après la récristallisation dans l'alcool méthylique l'acide hexahydro-benzile-éthyle-barbiturique se dépose en forme de lamelles, température de fusion 161—162° (non corrigée).

Analyse. Pesée 0.1272 g; 12.5 cm³ N (15° 762.2 mm).
$C_{13}H_{20}O_3N_2$. Trouvé: N = 11.41%.
Calculé: N = 11.11%.

Acide hexahydro-benzile-propyle-barbiturique

Dans un tube soudé on chauffe à une température de 90—95° pendant 4 heures 4 g d'éther hexahydro-benzile-propyle-malonique, 0.8 g d'urée et 20 cm³ d'une solution contenant 5% d'éthylate de sodium. A la fin de la réaction on met le contenu dans le ballon Wurz et on fait évaporer l'alcool aussi complètement que possible. Il en résulte une huile épaisse se solidifiant rapidement après avoir été lavée à plusieurs reprises par l'éther absolu. Le sel de sodium est dissout dans l'eau. Après que la solution aqueuse devient acidulée donnant une réaction acide indiquée par le congo, l'acide hexahydro-benzile-propyle barbiturique se dépose en cristaux. Pour l'obtention de l'acide à l'état pur, on dissout le dépôt en le chauffant dans une petite quantité d'alcool méthylique en y ajoutant de l'eau, il faut toutefois éviter un superflu, afin que l'acide ne se dépose en une masse huileuse se solidifiant rapidement. Après un court réchauffage, lorsque la solution se refroidie, des aiguilles très bien formées se déposent, température de fision 173—174° (non corrigée).

Analyse. Pesée 0.1360 g, 12.66 cm³ N (21°.5, 751.2 mm).
$C_{14}H_{22}O_3N_2$. Trouvé: N = 10.44%.
Calculé: N = 10.53%.

Acide hexahydro-benzile-isobutyle-barbiturique

Dans un tube soudé on chauffe, à une température de 90—95°, pendant 3 heures 6 g d'éther hexahydro-benzile-isobutyle-malonique (fraichement distillé), 3 g d'urée et 25 cm³ d'une solution contenant 5% d'éthylate de sodium.

Par la suite on procède de même qu'avec les acides dont nous avons donné la description ci-dessus. Pour obtenir l'acide, on récristallise le produit brut dans une solution aqueuse contenant 50% d'alcool. Le produit se cristallise en forme d'aiguilles; température de fusion 209°.

Analyse. Pesée 0.1268 g; 11.4 cm³ N (19°.5, 755.5 mm).
$C_{15}H_{24}O_3N_2$. Trouvée: N = 10.27%.
Calculé: N = 10%.

Acide hexahydro-benzile-isoamyle-barbiturique

Dans un tube soudé on chauffe, à une température de 90—95°, pendant 3 heures, 5 g d'éther hexahydro-benzile-isoamyle-malonique (fraîchement distillé), 2.5 g d'urée et 25 cm³ d'une solution contenant 5 % d'éthylate de sodium.

Par la suite on procède de même qu'avec les acides dont nous avons donné la description ci-dessus.

Après récristallisation dans une solution d'alcool éthylique et d'eau, le produit se dépose en forme d'aiguilles, température de fusion 184°.

Analyse. Pesée 0.1242 g; 10.1 cm³ N (18°,764 mm).
$C_{16}H_{26}O_3N_2$. Trouvé: N = 9.6%.
Calculé: N = 9.53%.

Institut d'étude et de synthèse de matières végétales et animales;
Académie des Sciences de l'URSS. Moscou.

Manuscrit reçu le 3. X. 1937.

LITTÉRATURE CITÉE

[1] E. Fischer u. Mering, Therapie d. Gegenwart, **3**, 97 (1903). [2] A. Дох, Z. Joder, J. Am. Ch. Soc., **44**, 1141 (1922). [3] Macht, J. Pharmacol., **11**, 176 (1918).

Comptes Rendus (Doklady) de l'Académie des Sciences de l'URSS
1937. Volume XVII, № 9

GEOLOGY

ON CIMMERIAN DEPOSITS IN SKADOVSK DISTRICT, UKRAINIAN SSR

By K. I. MAKOV

(Communicated by A. A. Borissiak, Member of the Academy, 3. XI. 1937)

Up to quite lately, the gas-bearing area immediately adjoining the Azov Sea coast (Priazovski district) was regarded as the westernmost boundary, of the Cimmerian sea in the Ukrainian SSR. The bore holes sunk in this district by M. N. Puchtinski with a view to prospecting the gasfield revealed yellow-green iron sandstones containing a rich Cimmerian fauna. Ten years ago it was mentioned by P. A. Dvoičenko that Cimmerian fauna had been encountered by him at Zburievka, near the river Dniepr. Later L. G. Tkačuk has discovered a Cimmerian shell at the same place in alluvial deposits that makes one suppose this fauna is to be found in a secondary deposit. This year I have succeeded in discovering the Cimmerian fauna in Skadovsk district, i. e. a considerable distance west of Priazovski district.

In 1936, a borehole was sunk here on land owned by the «Luch» Farming Association, Lenindorf village soviet, at a distance of 25—30 km north-west of Skadovsk. Above the meotic deposits, this borehole encountered a fine-grained, cemented yellowish-grey sand containing a large quantity of faunal remains in a good state of preservation. Among these shells V. G. Bondarčuk has identified the following forms: *Prosodacna macrodon* Desh.—2 specimens, *Dreissensia rostriformis* Desh.—many, *Dr.* cf. *angusta* Rouss., *Hydrodia* sp., *Bithinia* aff. *cyclostoma* Rouss., *Pantiapea* sp., *Limnocardium squamulosum* Desh., *Monodacna* sp. (cf. *pharnaci* Schw.), *Arcicardium* sp.

There is thus no doubt as to the Cimmerian age of these deposits.

The writer has also found Cimmerian *Prosodacna macrodon* Desh. and *Dreissensia* sp. in the samples drawn from the borehole sunk in Skadovsk. As these shells, however, are greatly obliterated, it is to be supposed that they occurred in a secondary deposit.

Finally, a third borehole sunk in the village of Karga, 12 km east of Skadovsk, struck no Cimmerian; the Pontic limestones are here overlaid by a thick series of sands with an abundant fauna, amongst which V. G. Bondarčuk has identified the following forms: *Dreissensia polymorpha* Pal., *Didacna* sp., *Dreissensia caspia* Eichw. This proves these sands to be of Old-Euxine age.

In the region around Karga village, the Cimmerian is thus found to have been washed away; nor has any been discovered at Khorly.

It may be seen from the above that: 1) Cimmerian strata in situ have been encountered in the Skadovsk district, which would suggest that the Cimmerian Sea extended much farther west than formerly supposed, reaching Ska-

dovsk district; 2) The Cimmerian deposits occur here as an island that has escaped destruction by scouring. As far as this may be inferred from the materials available, the Cimmerian had been washed off by a powerful stream (a river) flowing from east-north-east to west-south-west and having its mouth somewhere in the neighbourhood of Karkinit Bay.

Received
1. XI. 1937.

Comptes Rendus (Doklady) de l'Académie des Sciences de l'URSS
1937. Volume XVII, № 9

GENETICS

ON SOME INTERSPECIFIC CROSSINGS IN THE GENUS *GOSSYPIUM*

By N. N. KONSTANTINOV

(Communicated by N. I. Vavilov, Member of the Academy, 13. IX. 1937)

The method of hybridization in recent investigations on the taxonomy of plants is acquiring greater prominence, since the ability, or lack of it, of various forms to cross with each other, the behaviour of hybrids of F_1 -generation, the way of splitting in subsequent generations,—may aid in solving problems of phylogeny, taxonomy and so on.

In applying hybridization as an effective method in systematic research, certain factors must be kept in mind, which have a bearing on this process.

As an example we may cite Müntzig's (1) work on *Galeopsis*. Having crossed the species *Galeopsis pubescens* and *speciosa* ($2n=16$), he obtained a plant very unlike its parental forms, which was proved to be a triploid ($2n=24$). Crossing the hybrid with its parental form, *G. pubescens*, gave as a result a tetraploid ($2n=32$) identical with *G. Tetrahit* a species found in nature (one of the first cases of obtaining a Linnean species by synthetic means). The resulting hybrid, as well as the natural form of *G. Tetrahit*, would not cross with the parental forms. It follows that the ability of various forms to cross and the general character of hybrids, are not always a reliable criterion in solving problems of phylogeny, genotypical proximity, etc. *G. Tertahit*, which is by origin nearly related to the species *G. pubescens* and *G. speciosa*, behaves as a distant species, when an attempt is made to cross it with them.

The method of hybridization is extensively applied in the taxonomy of the genus *Gossypium*. Dr. Harland (2) in his latest paper on the classification of *Gossypium* comes to the conclusion that certain species of *Gossypium*, such as *Gos. Kirkii* M. Mast, should be excluded from this genus, since they neither graft nor cross with any cotton plant.

In connection with this problem, it is of interest to examine the results obtained in experiments on the species *Gossypium lanceoeforme* Miers. (*Thurberia thespesioides* Asa Gray)—a wild cotton plant commonly found in California and Arizona. In older classifications of *Gossypium*, based chiefly on morphological factors [Watt (3), Zaitsev (4)], this species was excluded from the genus, as it possesses a series of peculiar morphological features. *Gos. lanceoeforme* M. has lanceolate bracts, hairy boll folds, seeds, with an extremely hard, horny capsule*, etc. The species possesses peculiar biological

* In our conditions, the seeds of *G. lanceoeforme* M. would germinate only after having been treated by concentrated sulphuric acid or scraped by emery-paper.

properties as well. For instance, *Gos. lanceoeforme* M. is distinguished by a very short period of ripening of its bolls. The bolls of cultivated cotton plants ripen on the 45th day or later after flowering, whereas those of the *Gos. lanceoeforme* M. ripen on the 30th day. Originating from the comparatively northern areas of the cotton belt of America, it is further distinguished by its high resistance to frost.

As we had the opportunity to observe, this species of the cotton plant is able to withstand a temperature of —5°, —6° C. Its hardiness is further demonstrated by the fact, that its roots, if covered by soil, may last through the winter and give new shoots in the spring.

In this way, the species *Gos. lanceoeforme* M. is of interest, not only from the point of view of classification, but offers as well extensive possibilities for practical selection by means of hybridization. It is natural, therefore, that this species has lately attracted the attention of geneticists and selectionists, and that attempts are being made to cross it with other species of the genus *Gossypium*. The first results in this were obtained in 1931 by Harland and Atteck (5), who crossed it with *Gos. Stocksii* M. Mast, *Gos. Davidsonii* Kellog and two new-world forms (*Gos. barbadense* L.), and gave the species its present-day name. In 1935 Webber (6) succeeded in making the following crossings: *Gos. hirsutum* L.×*Thurberia thespesioides* Asa Gray and *Thurberia thespesioides* Asa Gray×*G. Sturtii* F. v. M.

The author notes that the hybrids of F_1-generation were sterile, and those of the first combination perished soon after the cotyledon stage.

Systematic research on the hybridization of *Gos. lanceoeforme* Miers. with other species of *Gossypium*, was begun by us in 1935. A large variety of forms from a world-wide collection of cotton plants was included in the process of crossing with this species. The results are given in the following table:

Nos	Combination (♀×♂)	Number of crossings made	Number of bolls obtained	% of bolls obtained
1	*G. barbadense* L. (*n*=26)×*G. lanceoeforme* M. (*n*=13)	272	12	4.4
2	*G. peruvianum* Cav. (*n*=26)×*G. lanceoeforme* M. (*n*=13)	54	1	1.8
3	*G. hirsutum* L. (*n*=26)×*G. lanceoeforme* M. (*n*=13)	276	12	4.3
4	*G. arboreum* L. (*n*=13)×*G. lanceoeforme* M. (*n*=13)	284	24	8.5
	Total	886	49	5.6

As can be seen from this table, the largest percentage of successful crossings falls on the last combination, viz. *G. arboreum* L. *G. lanceoeforme* Miers.

It was noted that the succes of the process depended not only on the choice of the species, but also on the selection of the particular variety, race and line of the species concerned. So, for example, it was observed that *Gos. lanceoeforme* Miers. could best be crossed with the variety *Gos. barbadense* L.-v. *maritima* Watt.; among the various *Gos. hirsutum* L. lines which took part in the crossings, the most successful crossings were obtained in the cases where line No. 1101 was used on the maternal side, etc. This appears to be due to certain biological peculiarities of the forms in question.

All of the bolls obtained from the above-mentioned crossings were found to be abnormal in development and in some cases bore unripe seeds; the majority, however, produced 2—3 seeds of normal appearance (altogether there were obtained about 100 seeds). The seeds germinated with difficulty, and we succeeded in obtaining but 40 hybrids from all of the above-mentioned combinations of crossings.

It is of interest to note the behaviour of hybrids of F_1-generation.

The hybrid resulting from the crossing *Gos. barbadense* L. × *Gos. lanceoeforme* M. was distinguished by vigorous growth. At the end of the vegetation period, some plants reached a height of 2.5 m. The morphological properties of the hybrid place it in an intermediate position between its parental forms.

The hybrid obtained by crossing *Gos. peruvianum* Cav. with *Gos. lanceoeforme* M. formed an almost complete analogy with the above-mentioned case. It only differed in having a slightly hairy stem and leaves, whereas those of the hybrids of the first combination were smooth.

This fact seems to justify the joining of the species *Gos. barbadense* L. and *Gos. peruvianum* Cav. The hybrids of both of the above-mentioned combinations were favourably affected by shortening of the day length.

Somewhat different was the behaviour of the hybrids obtained in crossing *Gos. arboreum* L. (v. *neglecta* Watt.) with *Gos. lanceoeforme* Miers. In contrast with the hybrids of the first combination, these hybrids were recorded to have a retarded growth of the stem. The hybrids possessed morphological features which place them in an intermediate position between their parental forms.

Entirely peculiar was the behaviour of the majority of the hybrids of the combination *Gos. hirsutum* L. × *Gos. lanceoeforme* Miers. As mentioned above, Webber in his paper comes to the conclusion that the hybrids of this combination perish soon after the cotyledon stage. In fact, the hybrid plants of the combinations which produced a maximum of hybrid seeds—line No. 1101 × *Gos. lanceoeforme* M. and line No. 1602 × *Gos. lanceoeforme* M. would not thrive in natural conditions. The plants were distinguished by a very retarded growth and, in the majority of cases, perished in 1—2 months after germination. A few plants survived till autumn, and one specimen, having lasted through the winter in a grass-plot of the hot-house, kept on growing the second year.

The cycle of development of these hybrids was as follows. Having reached a height of 5—6 cm, the plants ceased growing. The plants bore 2—3 undersized leaves, weakly growing; the leaves fell off without reaching their normal size and were gradually replaced by others. The plants possessed several anomalies in the development of their root system, were observed to have a retarded growth of the main stem, the latter being but slightly ramified, and the vital absorbing part of its root system was confined to small areas of the endings of the main and lateral roots.

Consequently, the behaviour of the bybrids of the two above-mentioned combinations of crossings of *Gos. hirsutum* L. with *Gos. lanceoeforme* Miers., coincides with the conclusions of Webber.

The third combination of the same series of crossings gave somewhat different results. In this case, the line *Gos. hirsutum* L. No. 9875 (derived from sample No. 01685 received from Bulgaria) was used on the maternal side. This line represents a *Gos. hirsutum* L. of an early variety cultivated in Bulgaria. By crossing the line No. 9875 with *Gos. lanceoeforme* Miers., a hybrid was produced which, although distinguished by a somewhat retarded growth, did not possess any of the above-mentioned anomalies. By the end of the season, the plant reached a height of 50 cm and developed normal leaves and flowers.

The morphological properties of the hybrid were likewise intermediate. All of the described hybrids were found to be self-sterile during the first year of development.

The results of the above described crossings point to the conclusion that the inclusion of the species *Thurberia thespesioides* Asa Cray in the genus *Gossypium* is correct. More complicated is the determination of the exact place of the species in the *Gossypium* system by the method of hybridization. In fact, the behaviour of hybrids of F_1-generation from the crossings of *Gos. barbadense*, *Gos. peruvianum* Cav., and *Gos. arboreum* L. with the said species, do not show any essential differences, whereas the maternal forms differ both in their chromosome numbers and their distribution.

The problem becomes still more complicated when one attempts to analyse the result of crossing *Gos. hirsutum* L. with *Thurberia thespesioides* Asa Gray.

If we limit the discussion to the results of crossing the species in question obtained by Webber, we may then suggest the inclusion of the species *Thurberia thespesioides* Asa Gray in the Genus *Gossypium*. Evidently, the result obtained by Webber is but a special case, and does not justify any generalizations on the entire species *Gos. hirsutum* L.

The above-mentioned facts lead to the following conclusions:

1. Hybridization is one of the essential links in the research work of the taxonomist, but it may only be made use of collaterally with other methods of systematic research.

2. In hybridological analysis, it is necessary to effect crossings on a vast scale, including the greatest possible variety of species and races. Otherwise one may come to erroneous conclusions.

Section of Botany
Central Plant Breeding Station.
All-Union Cotton Institute.
Tashkent.

Received
13. IX. 1937.

REFERENCES

[1] Müntzig, Hereditas, VI, H. 2 (1930). [2] S. C. Harland, Bibliographia Genetica, IX (1932). [3] G. Watt, The Wild and Cultivated Cotton Plants of the World (1907). [4] Г. С. Зайцев, Тр. по прикл. бот., ген. и сел., XVIII, вып. 1 (1927—1928). [5] S. C. Harland a. A. S. Atteck, Americ. Nature, **65** (1931). [6] Webber, Journ. of Agric. Res. **51**, № 12 (1935).

Comptes Rendus (Doklady) de l'Académie des Sciences de l'URSS
1937. Volume XVII, № 9

GENETICS

WHEAT-QUITCH AMPHIDIPLOIDS

By V. A. HIŽNIAK

(Communicated by N. I. Vavilov, Member of the Academy, 1. XI. 1937)

After studying wheat-quitch hybrids we have succeeded in obtaining wheat-quitch amphidiploids, highly promising plants both from the theoretical and the practical point of view.

We had formerly described the process of the evolution of new forms in wheat quitch hybrids derived from crossings of wheats with *Agropyrum intermedium* in the case of pollination with F_1 pollen of either quitch or wheat; we stated on that occasion that this gave rise to the formation on sesquidiploids or triple hybrids of both the wheat and the quitch type (1).

In this paper, we may state that the self-pollination of the first generation of wheat-quitch hybrids obtained by crossing wheats with *A. intermedium* ($n=21$) gives as a rule amphidiploids. Amphidiploids have been obtained in the following combinations: *Triticum durum* × *Agropyrum intermedium*, *Tr. dicoccum* × *Agr. intermedium*, *Tr. turgidum* × *Agr. intermedium*, *Tr. polonicum* × *Agr. intermedium*, *Tr. persicum* × *Agr. intermedium*.

The amphidiploids of all the above combinations showed, as was to be expected, 70 chromosome to the soma.

The reduction division proceeds regularly enough, 33—35 bivalents and 0—4 univalents being observed, as a rule, in metaphase I; polyvalents are sometimes formed. The pollen is 95—97% fertile. The coefficient of fertility of the amphidiploids is 1.6—2, i. e. 1.6 to 2 grains are formed on every ear.

The amphidiploid plants are sturdy perennials showing a luxuriant growth and flowering openly; their fertility is not decreased by self-pollination. The ear of an amphidiploid is practically undistinguishable from an F_1 ear of the combination concerned. The grain of an amphidiploid has an absolute weight of 16—20 g. These plants are immune to rust and winter well in the climatic conditions of Krasnodar.

It is our considered opinion that wheat-quitch amphidiploids have an enormous practical value as a perennial cereals grass. The dense, sturdy bushes, with an abundant foliage rising up to the very ears, the ability to grow very fast after mowing, a luxuriant growth, the slenderness of the stems, the considerable height of the plants (120 cm) and their 100% fertility, testify to the high qualities of wheat-quitch amphidiploids as a fodder grass. The absence of rhizomes, a well developed fibrous root system, capable of restoring the structure and the fertility of the soil, the large size of the grains, these are the most valuable properties of wheat-quitch amphidiploids as a new perennial cereal grass.

The high absolute weight of the grain of wheat-quitch amphidiploids makes it possible to use them as fodder for cattle.

In 1937, we sow the amphidiploid over an area of 1 hectare, and consider that it is possible to have next year as many as 30—50 hectares sown with the amphidiploid.

We are of opinion that the wheat-quitch amphidiploid can be grown throughout the whole of the vast area of distribution of winter wheat.

Now that a regular rotation of grass crops is being introduced, the requirements of farming in a productive perennial cereal grass are enormous. The South of the USSR, however, has no such productive cereal grass. The wheat-quitch amphidiploid may supply a favourable solution to this important economic problem.

State Plant Breeding Station.
Krasnodar

Received
1. XI. 1937.

REFERENCES

[1] Селекция и семеноводство, № 62, 1936.

Comptes Rendus (Doklady) de l'Académie des Sciences de l'URSS
1937. Volume XVII, № 9

PLANT PHYSIOLOGY

NUTRITIVE MEDIA FOR SUGAR BEET IN SAND AND WATER CULTURES

By F. UKRADYGA and A. OLEKSIJUK

(Communicated by A. A. Richter, Member of the Academy, 2. XI. 1937)

The development of so-called nutritive media, the absence of which has so far hindered the study of plant growth, is a very important section of agrochemistry and physiology.

Attempts to make up such media have been made for a long time past, but Hellriegel alone succeeded in evolving a medium whereon plants developed quite normally. It is also to him that we owe the methods of raising sand cultures. The medium made up by Hellriegel is meant mainly for cereals. Subsequent work done in this connection also had in view those same cultures. In this country, scant attention has been paid to this problem, except by the Laboratory of Prof. Pryanišnikov. The work done in the last named laboratory since 1899 also deals with cereals. Great differences are known to exist between different plants as to composition and nutritive requirements. This hold good for a number of plants used for industrial purposes, and, more especially, for sugar beet.

This is why none of the attempts to adapt for sugar beet the nutritive media used for cereals have been successful.

It is, as a matter of fact, impossible to prepare a medium of universal application.

The absence of normal nutritive media suitable for beet root has induced us to eneavour to evolve such media.

In making up a nutritive medium for beet root we were guided by the following considerations:

1) To make up a nutritive medium of such a nature that both the total amount of nutritive elements contained therein and their ratios should ensure the normal growth of the plant and a plentiful crop.

2) The medium should be a balanced one, it must have an optimal and stable concentration of hydrogen ions and should contain no unnecessary salts.

The composition of the nutritive medium was such as to correspond to the data available on the assimilation of nutrient substances by the beet. The results obtained by Mironovskaja, Nemerchanskaja and other experimental stations were used for the purpose, in addition to the well known works of Krüger and Wimmer, and of Werenbourg experimental station for the 1905—1925 period. Taking into consideration the coefficient of assimilation of nutrient substances by the beet, we calculated the amount of the

latter thatis required to produce a yield of 800 g (600 g root plus 200 g leaf) and determined the forms in which such substances should be introduced (Table 1).

Table 1

Nos	Description of salt	Grams per 16 kg sand
1	$Ca(NO_3)_2 \cdot 4H_2O$	18.38
2	K_2HPO_4	4.80
3	KCl	1.60
4	KNO_3	3.00
5	$NaNo_3$	4.91
6	$MgSO_4 \cdot 7\ H_2O$	8.10
7	$CaCO_3$	6.00
8	$Fe_2(SO_4)_3$	1.20
9	H_3BO_3	0.045
10	$MnSO_4 \cdot 5\ H_2O$	0.24
11	$Al_2(SO_4)_3 \cdot 18\ H_2O$	0.10

Table 2

Nos	Media	Weight in grams			Root to leaf ratio	Sugar percentage
		Whole plant	Root	Leaf		
1	New medium	790	600	190	3.1	19.2
2	Malutitzki	524	323	201	1.5	16.0
3	Belousov	443	347	95	3.5	20.4
4	Tolens	417	287	130	2.2	19.2
5	Somer-Lipman	373	316	57	5.5	19.3
6	Filippovski 10	227	167	60	2.9	19.8
7	» 20	243	175	73	2.4	20.0
8	Rozhdestvenski	466	277	189	1.4	17.2
9	Stoklaza	230	156	74	2.1	19.6

The quantitative ratios of the elements of nutrition are as follows (in grams per 16 kg of sand):

$$N : P : K : Na : Ca : Mg = 3.39 : 0.85 : 4.15 : 1.35 : 10 : 0.76.$$

The new medium was compared with other media that were available in 1936. The experiment began on May 8, and the crop was collected on September 7.

The yield values appearing in Table 2 show clearly that not all the media can be used successfully in growing beet. As shown by comparative tests, the best results in growing beet in sand were obtained with the new medium, Malutitski medium, and Belousov medium. The highest yield was obtained with the new medium (600 g root and 190 g leaf). Our salt estimates proved correct, and, in some instances, the weight of the roots was even slightly higher. The percentage of sugar is a fairly high one, varying from 16 to 20.4% according to mixture.

The ratio of root weight to leaf weight shows that not all of the plants collected were of the same degree of ripeness, the latter being apparently greatly dependent on the elements of nutrition in the media.

In order to ascertain the effect of potassium on the development of sugar beet, some experiments with water cultures had also to be made. The absence

of suitable media for growing beet either in sand or in water induced us to endeavour to evolve a medium for water cultures of beet. The total salt content in grams per litre of distilled water was as shown below (Table 3).

Table 3

Nos	Description of salt	Medium of the year 1936	Medium of the year 1935
1	$Ca(NO_3)_2 \cdot 4H_2O$	1.77	1.18
2	$NaNO_3$	0.85	0.50
3	K_2HPO_4	0.58	0.50
4	KCl	0.37	0.20
5	$MgSO_4 \cdot 7H_2O$	0.90	0.61
6	$Fe_2(SO_4)_3$	0.15	0.80
7	$MnSO_4 \cdot 5H_2O$	0.025	0.004
8	H_3BO_3	0.005	0.005
9	$Al_2(SO_4)_3 \cdot 18H_2O$	0.013	—
10	$CaCO_3$	0.1	0.1

The new media were compared with the following known media: Belousov, Hellriegel, Tolens, Knoop. Krone, Cincadse, Somer-Lipmann, Mase, Naftal. The experiments were made in three sets, in 4-litre vessels; the experiments began on May 13, and the crop was collected on October 16. The first 2—3 solutions used contained only half the amounts of salts shown above, but the media used in subsequent periods contained the full amount of salts. A change of solutions took place every 15—20 days.

Table 4

Nos	Media	Weight in grams			Root to leaf ratio	Sugar percentage
		Whole plant	Root	Leaf		
1	New medium of 1936	715	380	335	1.13	17.0
2	» » » » (half)	676	405	271	1.49	17.7
3	» » of 1935	827	490	337	1.45	18.0
4	Belousov	455	292	163	1.79	18.3
5	Tolens	505	300	205	1.46	16.9
6	Krone	391	241	150	1.60	17.6
7	Somer-Lipmann	394	212	182	1.16	13.4
8	Cincadse	308	165	143	1.15	18.6
9	Knoop	288	152	136	1.12	17.2
10	Hellriegel	137	92	45	2.04	18.5
11	Mase	120	62	8	1.04	13.4
12	Naftal	38	23	15	1.53	11.4

The above data (Table 4) show that the best media for sugar beet are: the new media of 1935 and 1936, Belousov and Tolens. All the other media are far below these in point of yield.

In all the media, the weight of the roots is greater than that of leaf, no constant ratio betwen them being, however, observable. With the exception of Mase and Naftal media, the sugar content is rather high, varying from 16.9 to 18.6%. The water cultures were unfavourably affected by the very hot summer of 1936. The small vessels became superheated, the temperature being frequently as high as 30—35°C. The plants growing in larger vessels were less affected by superheating. As a general rule, the volume of the vessels is of the utmost importance in growing beetroot in water, apart from salt content, the kinds of salts, the ratios of the elements of nutrition, pH, etc.

In 4—5-litre vessel the new media produced roots of a total weight of over 700 g, as against roots of over 1 kg obtained by Belousov with his own medium in 18-litre vessels.

All-Union Institute
of Sugar Industry. Kiev.

Received
26. X. 1937.

Comptes Rendus (Doklady) de l'Académie des Sciences de l'URSS
1937. Volume XVII, № 9

PLANT ANATOMY

ON THE MOSAIC OF WHEAT ENDOSPERM

By V. G. ALEKSANDROV and O. G. ALEKSANDROVA

(Communicated by N. I. Vavilov, Member of the Academy, 23. X. 1937)

Due attention has not been paid up to the present to the structure of the endosperm of cereal grains owing to the erroneous idea that the endosperm tissue of all the representatives of one genus, at least, is identical in structure. We believe that defects in the technique of investigation are to a considerable degree responsible for the existence of such an idea regarding the structure of endosperm. A criticism of these methods and a description of a new and more rational technique was given in our joint paper (1). The new technique consists in examining grain sections in solutions of glucose or saccharose for in such a medium starch and protein, the components of endosperm structure, are best preserved in their natural state.

The investigations of the authors were mainly centred on a study of the structure of the grain of various wheats. 25 per cent is the best concentration of glucosa for 14-chromosome wheats, 10 per cent for 28-chromosome ones, and 5 per cent for 42-chromosome wheats. The application of the above-mentioned new method has enabled the authors to establish the fact that each wheat species possesses its own peculiar structure of the contents of the endosperm cells and, what it is more, that the differences in the structure of the endosperm of various wheats are so peculiar that in a number of cases it is even possible to distinguish varieties.

In their natural condition, the starch and protein in the wheat endosperm, as in that of all other cereals, are found to occur in definite proportions to each other. The existence of a regular relationship is especially noticeable as regards the starch of every systematical wheat individual. In the endosperm of wheat we find starch of several types, differing both in shape (oval, round polyhedral) and size (coarse-grained, medium-sized, fine-grained). Every wheat form is always distinguished by its own typical relationship in the arrangement of the starch grains, its endosperm possesses its peculiar mosaic pattern. It is, therefore, possible to identify wheats by their endosperm mosaic.

What are the structural principles of the mosaic pattern in the wheat endosperm?

Let us examine four cases of the structure of the central part of the endosperm tissue of 42-chromosome wheats, one case of 28-chromosome ones, and one case of 14-chromosome ones. Fragments of these endosperm tissues are shown in the figures.

Fragment 1 shows a part of the endosperm tissue of *Tr. spelta* L. from Austria (Tyrol). The endosperm mosaic of this form of wheat is distinguished

by the comparatively small amount of coarse-grained (plastid) starch. The grains of plastid starch are fairly far apart and the interspaces between them are filled with fine-grained (chondriosomal) starch. The grains of the latter present as regards size, a marked contrast to those of the plastid starch. The grains of chondriosomal starch are very closely set together so that the intercalations of protein between them are insignificant. In the central part of the endosperm tissue, the mass of accumulated chondriosomal starch is obviously greater than that of plastid starch. The chondriosomal starch of spelt, as of all other wheats is either polyhedral or round. The mutual arrangement of grains of these two forms, however, is specific for each mosaic type. In the case under consideration, the polyhedral grains considerably exceed the round ones in number. The latter are scattered locally, being fairly well

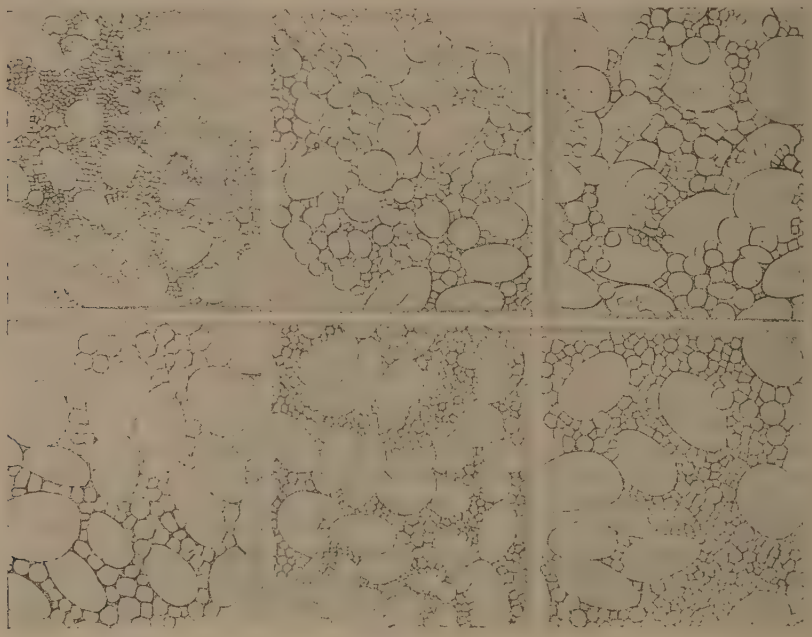


Fig. 1.

distinguishable against the background of the angular polyhedral grains. The mosaic seen in fragment 1 is thus distinguished by the large quantity of closely packed chondriosomal starch grains, exceeding in their mass that of plastid starch, the grains being mainly of a sharply angular shape.

Fragment 2 represents a portion of the endosperm tissue of a soft wheat of the standard variety Lutescens 0329 (v. *lutescens* Al.). As compared with fragment 1, the type of mosaic here is somewhat different. The number of grains of plastid starch is considerably higher per unit area of endosperm tissue. They are more closely set, and the interspaces between them, filled with chondriosomal starch, are not so large as in spelt. The chondriosomal starch is very fine-grained, polyhedral grains prevailing, though in their majority they are not sharply angular. This produces the impression of a looser arrangement of the chondriosomal starch grains. The intercalations of protein between the starch grains are, consequently larger. Thus, the mosaic of fragment 2 is distinguished by a greater amount of plastid starch

and a looser arrangement of the chondriosomal starch grains, these being mainly polyhedral but not angular.

Fragment 3 shows a part of the endosperm tissue of an endemic soft wheat of Afghanistan of the rigid wheat group (v. *erythrospermum* Körn). The amount of both plastid and chondriosomal starch is relatively small. Although polyhedral grains of chondriosomal starch exceed the round grains in number, the percentage content of the latter is noticeably higher than in the two preceding types of mosaic pattern. Thus, the amount of polyhedral grains of chondriosomal starch is especially low in the case under consideration. Moreover, all the chondriosomal starch is relatively coarse-grained, so that large spaces filled with protein are of frequent occurrence. In brief, the mosaic of fragment 3 is distinguished by a small amount of relatively coarse-grained chondriosomal starch and the presence of large interspaces filled with protein.

Fragment 4 shows a portion of the endosperm tissue of the standard Ukrainka variety (v. *erythrospermum* Körn). Besides the presence of large interspaces filled with protein, this case of mosaic pattern is distinguished by a great diversity in size of the chondriosomal starch. In the preceding cases of mosaic pattern the chondriosomal starch grains mainly differed in the shape of the individual grains. In the case in question, one may distinguish at least three sizes of chondriosomal starch grains, both of the polyhedral and the round ones. The most important fact is that in this case the difference in size of the starch grains is much greater and more distinct than in the other cases examined by us. The polyhedral starch grains, especially the finest of them, are clustered into small groups and seem to be separated from the round grains, thus making the prevalence of the latter more pronounced. The round grains are the most loosely arranged.

The prevalence of the round chondriosomal starch grains over the polyhedral ones is still more marked in fragment 5, which shows a portion of the endosperm tissue of a spelt originating from Spain (*Tr. dicoccum* (Shrank) Schübl., a 28-chromosome wheat). In a number of points of the endosperm tissue, the round grains form peculiar centres around which are grouped polyhedral starch grains, nowhere attaining the size of the round ones. On the other hand, among the large chondriosomal grains occur such grains as may be considered to be transitional in size to the plastid starch grain. The size of the round chondriosomal starch grains is as diverse as in the case of the Ukrainka endosperm. The size of the polyhedral grains is on the contrary quite uniform. This is the principal difference between the mosaic pattern of the endosperm tissue of the spelt and that of the Ukrainka. The mosaic of fragment 4 is thus distinguished by the diversity in size of both the round and the polyhedral chondriosomal starch grains, whilst the mosaic of fragment 5 is distinguished by the diversity of the round grains only. But the plastid starch grains in the mosaic of fragment 5 are generally much finer than those of fragment 4.

Fragment 6 represents a portion of the endosperm tissue of a cultural monococcal wheat from Spain (*Tr. monococcum* L. v. *vulgare* Körn., a 14-chromosome wheat). The mosaic of the endosperm tissue of this cultural monococcal wheat is distinguished by the extreme fineness of all the components of the mosaic and by the large amount of closely packed polyhedral chondriosomal starch. Round chondriosomal starch does not prevail over the polyhedral, same as in fragment 2, but here it is more diverse in size. The difference in size between the chondriosomal and the plastid starch grains is well marked.

The above described six cases of the mosaic pattern of the endosperm tissue of various cultivated wheats with a different chromosome number

clearly show that the mosaic, or the relationship between the components of the endosperm tissue, is a real phenomenon. In each of the various wheat forms it is possible to distinguish a definite mosaic type, which characterizes, to a certain extent, a number of grain qualities, e. g. glassiness and flouriness.

Received
23. X. 1937.

REFERENCES

[1] Александров и Александрова, Тр. по прикл. бот., ген. и сел., серия V A, № 2 (1936).

Comptes Rendus (Doklady) de l'Académie des Sciences de l'URSS
1937. Volume XVII, № 9

AGROPHYSIOLOGY

THE CONTINUOUS SUPPLY OF PLANTS WITH WATER AS AN INDISPENSABLE CONDITION OF HIGH YIELDS

By N. S. PETINOV and P. S. BELIKOV

(Communicated by A. A. Richter, Member of the Academy, 4. XI. 1937)

Academician Williams has stated in a few words the most important law of agriculture: «The simultaneous and continuous supply of plants with the maximum necessary quantity of water and food is the basis of high yields». The success of the Stakhanovites of agriculture, who have surpassed the old «norms» of crop yields, are essentially and indissolubly associated with the application of this principle.

The Institute of Plant Physiology of the Academy of Sciences has also been taking part during the last few years in the study of the irrigation problem of the Trans-Volga regions. In 1937, a series of experiments was carried out by the Institute on the irrigated field and in the greenhouse of the Kuibyshev Agricultural Institute (Kinel). The object of these experiments was to make a physiological analysis of the spring wheat plants grown under conditions of a continuous supply with water and food. In view of this the dynamics of the moisture and the nutrient substances in the soil as well as the physiological and chemical clanges occurring in the plant itself were continually studied during the whole period of growth of the experimental plants.

As there were rains almost daily during the whole of May, totalling 97. 1 millimetres of precipitation, we made the first watering on June 7, towards the end of the «tillering» stage. The second watering was made on June 27, at the beginning of the «earing» stage. The two waterings, which wetted the soil to a depth of 80 cm (600 cubic metres of water per hectare), guaranteed an optimal humidity in the sixty centimetre layer of soil (common carbonate chernozem) for the entire duration of the vegatative period. The humidity of the soil never dropped below 20 per cent, the unavailable moisture being 13.5%. Thus, two waterings ensure for the wheat a continuous water supply. In order to set up similar conditions of nutrition, ammonium nitrate was introduced before the first and the second waterings.

Table 1 gives the dynamics of the nitrates in the soil and shows that the introduction of ammonium nitrate on the eve of the waterings best satisfies the requirements of a continuous supply of the plant with nitrogen. In such a manner, the content of nitrates in the soil is maintaned at a relatively high level during the whole vegatative period. It is very probable that a further increase of their content in the soil may be obtained by applying such an introduction of fertilizer combined with a p r e-s o w i n g f e r t i z a t i o n. If the same quantity of ammonium nitrate be introduced all at one

time before sowing, the accumulation of nitrates proceeds very unsteadily. Their amount markedly increases during the first half of the vegetative period and falls during the second half to such an extent, that it becomes similar to that in unwatered and unfertilized plots.

Table 1

The Dynamics of Nitrates in the Soil of a Watered Plot of the Kuibyshev Agricultural Institute, 1937

(Determined after the method of Grandval-Lajou)

Plot	Depth in cm	Milligrams of NO_3 per kilogram of dry soil			
		5.VI	10.VI	25.VI	18.VII
Unwatered and unfertlized	0—20	23.9	36.5	16.5	13.5
	20—40	25.4	41.0	24.2	14.6
	40—60	113.5	49.6	22.3	6.6
Mean for the 60 cm layer		54.3	42.4	21.0	11.6
Watered but unfertilized	0—20	—	30.0	25.8	9.2
	20—40	—	26.6	23.2	9.3
	40—60	—	115.7	73.3	44.1
Mean		—	57.4	40.4	20.9
90 kg* of P_2O_5 and 90 kg* of N before sowing	0—20	89.6	227.9	30.2	12.6
	20—40	24.8	199.0	20.3	10.8
	40—60	113.5	146.9	22.8	8.5
Mean		76.0	191.2	24.4	10.6
90 kg of P_2O_5 before sowing	0—20	113.6	65.0	40.8	47.1
45 kg* of N » tillering	20—40	43.2	46.9	25.6	65.0
45 kg* of N » earing	40—60	153.9	48.4	65.8	16.2
Mean		103.6	53.4	44.1	42.7

Analogous results were obtained in the All-Union Institute of Grain Farming (Saratov) (4), but erroneous conclusions were drawn therefrom; the main (pre-sowing) fertilization was opposed to the supplementary one, whereas it should be combined with the latter. In the opinion of the workers of that Institute, spring wheat grown on dark chestnut soils *does not need* any supplementary supply of nitrogen when watered before the tillering stage, and the first introduction of nitrogenous fertilizers should be made before the first watering and before sowing (the italics are ours, P. and B.). Janšina bases her conclusions on the data concerning the energy of tillering and the yield. In her recent paper (4) she writes: «in the variant where the entire dose of fertilizer was introduced before sowing, the tillering was more vigorous before the beginning of earing (June 17). With the onset of the latter

* Referred to phosphorus and nitrogen per hectare.

stage, tillering decreases greatly. In the variants where the fertilizer is introduced in parts at two or three dates, it does not decrease in vigour almost right up to the end of the vegetative period». The author overlooks the fact that in the case of a single pre-sowing introduction of nitrogenous fertilizers, the tillering of spring wheat was more vigorous than when the fertilizers were introduced in parts. Therefore, such energetic tillering can be maintained till the end of the vegetative period by the additional introduction of nitrogenous fertilizers, i. e. by combining the pre-sowing fertilization with the introduction of fertilizers in parts. The same applies to the yield.

We find the following data in Janšina's paper (4):

1. Without fertilizers, 19.7 metric centners per hectare
2. $P_{90}N_{90}$ 25.5 » » » »
3. $P_{90}N_{60}+N_{30}$ 28.5 » » » »
4. $P_{90}N_{30}+N_{30}+N_{30}$ 26.4 » » » »

The above figures show that, in the case of one pre-sowing fertilization with nitrogenous fertilizers, the increase in the yield is 5.8 centners as compared with the control, while the difference between the yields obtained in the case of a single pre-sowing fertilization and a fertilization «by instalments» was in one case (two instalments) 3 centners and in the other (three instalments) 0.9 centners. These data show that the plant's need of nitrogen is greatest in the early periods—before the onset of earing. This does not contradict the possibility of obtaining higher yields by the application of supplementary feeding combined with a pre-sowing introduction of fertilizers.

The results of the investigations carried out by the Institute of Plant Physiology on the dynamics of nitrates in the soil (Table 1), as well as the determinations of assimilation (increased energy of assimilation before the onset of the earing stage, when the fertilizer is introduced before sowing, as compared with that when it is introduced «by instalments» before the 1st and 2nd watering), speak in favour of combining the fundamental pre-sowing fertilization with supplementary feedings.

Table 2

The Grain Yield of the Spring Wheat *Hordeiforme* 0189 on Irrigated and Fertilized Soil (in centners per hectare)

Nos	Description of experimental plots	Yields of grain	
		2 centners per hectare	In percentages
1	Control, unwatered and unfertilized	14.6±0.23	100%
2	Watered but not fertilized	27.2±0.41	186.3
3	Watered; fertilized with 90 kg P and 90 kg N before spring harrowing	45.3±0.37	310.3
4	Watered; fertilized with 90 kg P and 30 kg N before spring harrowing+30 kg N before 1st watering during tillering+30 kg N before 2nd watering at onset of earing	50.0±0.31	342.5
5	Watered; fertilized with 90 kg P before spring harrowing+45 kg N before 1st watering during tillering+45 kg N before 2nd watering at onset of earing	51.7±0.38	361.0

In conclusion we shall make a few observations concerning the existing «norms» of fertilization. Recipes with quite different dosages may be met with for approximately identical conditions. Thus, for instance, Čižov(3)

recommends to introduce from 30 to 90 kg of nitrogen, whilst Nikolaev (2) recommends to introduce up to 200 kg. At the same time, neither of these writers adduces arguments in favour of the norms in question. Bare recipes are given, no answer being supplied to the question of whether the dosage recommended ensures a continuous supply of the plant with food. High and steady yields will only then be ensured when instead of applying ready-made schemes and norms of watering expressed in terms of cubic metres per hectare, and norms of fertilization, expressed in terms of kilograms per hectare, both water and fertilizers will be supplied at the dates and in the quantities required by the plant. But this is possible only if throughout the development of the plant a record be kept of the quantity of food and water available to and necessary for the plant. The soil lives, same as the plant does, and therefore requires care based on dynamic and not static studies.

Conclusions

1. The continuous and simultaneous supply of plants with the maximum necessary quantity of water and food, as the condition of high yields, formulated by Academician V. R. Williams, had been ignored for a number of years by «scientists» defending the «theory of limits» of yield.

2. The experiments carried out by the Institute of Plant Physiology of the USSR Academy of Sciences in 1936 and 1937 under field conditions in the Trans-Volga regions, fully confirmed this fundamental law of scientific agriculture.

3. Under the conditions that prevailed in the summer of 1937 two waterings ensured a continuous supply of water for the wheat and, at the same time, caused a marked increase in the content of nitrates in the soil.

4. We consider that the yield of 51.7 centners per hectare obtained in the case of a continuous supply of water combined with supplementary feeding is insufficient and put it down to the small amount of mineral ferilizers introduced. 90 kilograms of nitrogenous fertilizers (referred to nitrogen) before sowing is not sufficient. The quantity should be increased at the expense of the supplementary feeding, to compensate for the marked drop in the content of nitrates observed to occur in the second half of the vegetative period.

5. The introduction of ammonium nitrate «by instalments» (supplementary feeding) causes a uniform accumulation of nitrates throughout the period of growth of the wheat.

6. The opposing of fertilization «by instalments» to pre-sowing fertilization, as made by the Institute of Grain Farming, is misleading and results from ideas whose holders used to recommend late waterings instead of a continuous water supply for the plant.

Timiriasev Institute of Plant Physiology.
Academy of Sciences of the USSR.
Moscow.

Received
4. XI. 1937.

REFERENCES

[1] В. Р. Вильямс, Социалист. реконструкц. с.-х., № 6 (1937). [2] И. Т. Николаев, Социалист. зернов. хоз-во, ВИЗХ. [3] Б. Чижов, Социалист. зернов. хоз-во, ВИЗХ, № 4 (1936). [4] М. Я. Яншина, Социалист. зернов. хоз-во, ВИЗХ, № 5 (1937).

Comptes Rendus (Doklady) de l'Académie des Sciences de l'URSS
1937. Volume XVII, № 9

DEVELOPMENT MECHANICS

THE PRODUCTION OF SUPERNUMERARY BALANCERS IN DEFICIENCY EXPERIMENTS IN TRITON EMBRYOS

By B. I. BALINSKIJ

(Communicated by I. I. Schmalhausen, Member of the Academy, 1. XI. 1937)

During numerous recent experiments there has been observed the development of supernumerary balancers in operated Triton embryos. The balancers appeared after the implantation of different killed and living tissues under the ectoderm (into the blastocoel) of young embryos. As the supernumerary balancers were usually situated near the grafts, their development was interpreted as the result of an «induction» ([1—5]). Raven states though that often the «induced» balancers do not lie in immediate contact with the graft but at some distance from it.

In spring 1937, I observed the appearance of supernumerary balancers in a series of experiments. (Series Uen), in which no additionary objects that could act as inductors were introduced into the embryo, but on the contrary, there were excised more or less great parts of the young embryo. In the late blastula or early gastrula stage I excised the major part of the future entoderm of the embryo, the region of the fore gut and partly the adjacent districts of the «Randzone» included. Eventually there developed deformed embryos, often with a derangement in the development of balancers. It was a surprise to find that there was not only a insufficiency of balancers , but there was also observed an unusually great number of them: 4 and even 11 balancers. It is interesting to compare this result with the result of another series of experiments (Series Gen), executed simultaneously, that consisted in the transplantation of different parts into the blastocoel of young gastrulae or late blastulae. In this series also there was observed the formation of supernumerary balancers (6 in one case). Supernumerary balancers appeared not only in such cases, in which the graft lay in the head region and could act as their inductor, but also in cases in which the graft lay in the posterior part of the embryo. I suppose that, in this series, the operation caused a derangement of the gastrulation (many embryos in this series showed exogastrulation in a greater or less degree) and this produced a deficiency in the ento-mesoderm comparable to that caused in the former series by direct excision.

Our data allow us to suppose that in the normal development there takes part some factor that prevents the development of more than two balancers, or that confines the balancer development to two small districts in which they normally occur. If this factor, which is apparently localized in the ento-mesoderm, be removed, the tendency for balancer development becomes

liberated, and supernumerary balancers are formed in regions adjacent to the normal balancers.

It is interesting to note that the supernumerary balancers are almost always situated in the form of a row or comb across the antero-ventral part of the head. This observation may serve to indicate the position of the material that possesses the tendency for balancer development.

If the above interpretation of the production of supernumerary balancers is correct, it would be necessary, when investigating the determination of parts in the vertebrate embryo, to take into account the possibility that besides the stimulating influences of parts (inductions) there exist interactions that limit the realization of certain formative processes.

Institute of Zoology and Biology.
Academy of Sciences of the Ukrainian SSR.

Received
1. XI. 1937.

REFERENCES

[1] J. S. Holtfreter, Roux' Arch., **132**, (1934a). [2] J. Holtfreter, Roux' Arch., **132** (1934b). [3] O. Mangold, Die Naturwiss. (1933). [4] C. P. Raven, Proc. Konink. Akad. van Wet. Amsterdam, **34** (1931). [5] C. P. Raven, Roux' Arch., **130** (1933).

IMPERIAL MYCOLOGICAL INSTITUTE
6 OCT 1938